

6.2 Master Theorem

Note that the cases do not cover all possibilities.

Lemma 5

Let $a \geq 1$, $b > 1$ and $\epsilon > 0$ denote constants. Consider the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) .$$

Case 1.

If $f(n) = \mathcal{O}(n^{\log_b(a)-\epsilon})$ then $T(n) = \Theta(n^{\log_b a})$.

Case 2.

If $f(n) = \Theta(n^{\log_b(a)} \log^k n)$ then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$,
 $k \geq 0$.

Case 3.

If $f(n) = \Omega(n^{\log_b(a)+\epsilon})$ and for sufficiently large n
 $af\left(\frac{n}{b}\right) \leq cf(n)$ for some constant $c < 1$ then $T(n) = \Theta(f(n))$.

6.2 Master Theorem

We prove the Master Theorem for the case that n is of the form b^ℓ , and we assume that the non-recursive case occurs for problem size 1 and incurs cost 1.

The Recursion Tree

The running time of a recursive algorithm can be visualized by a recursion tree:

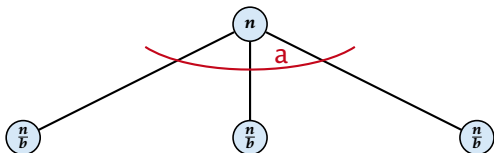
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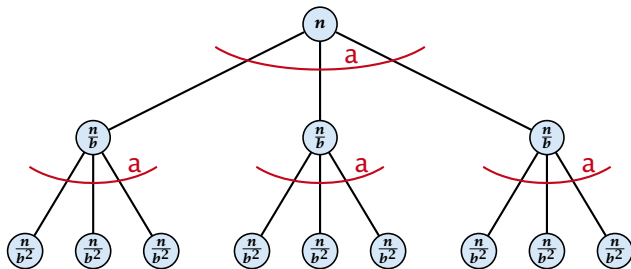
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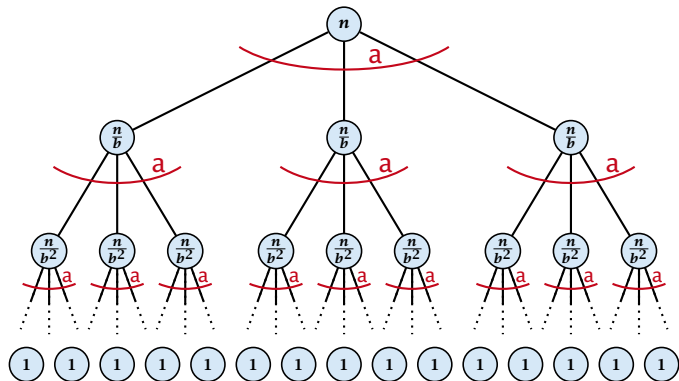
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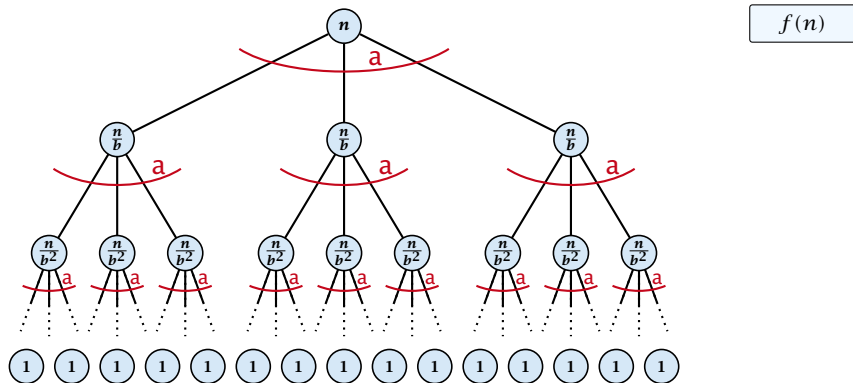
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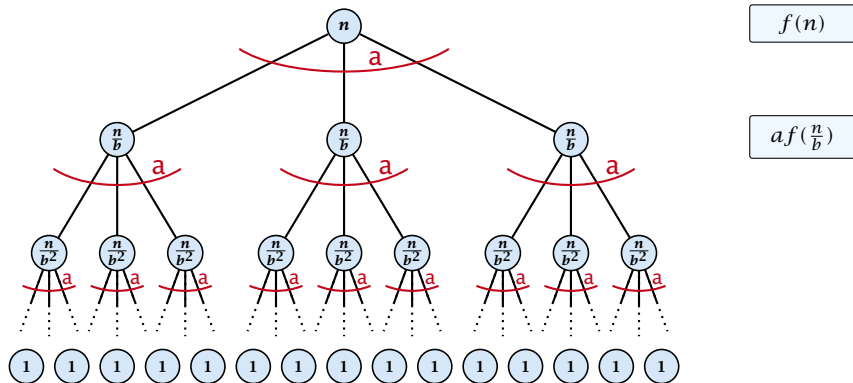
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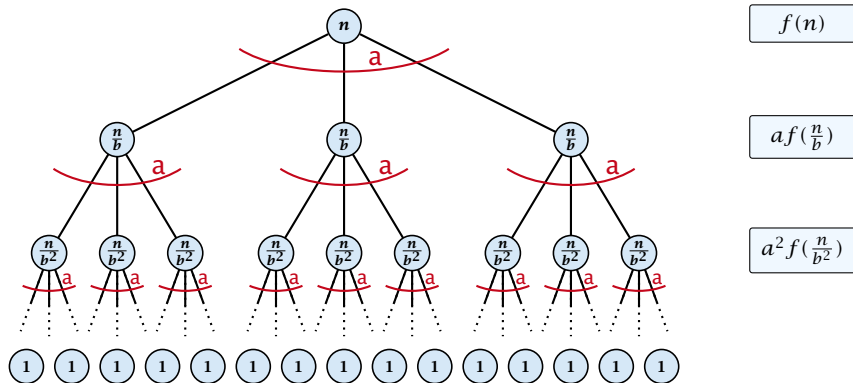
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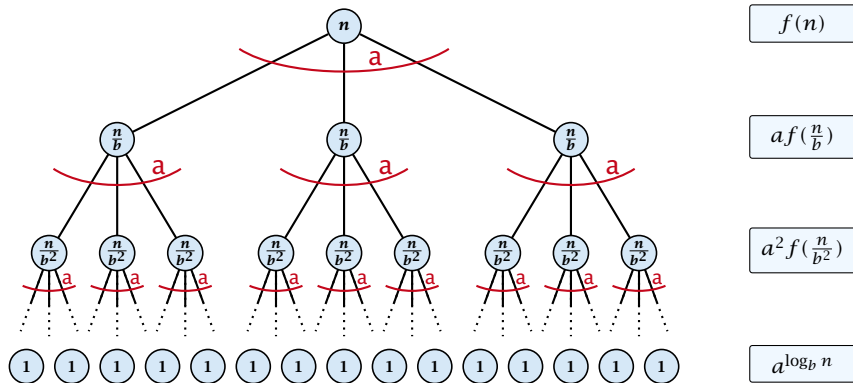
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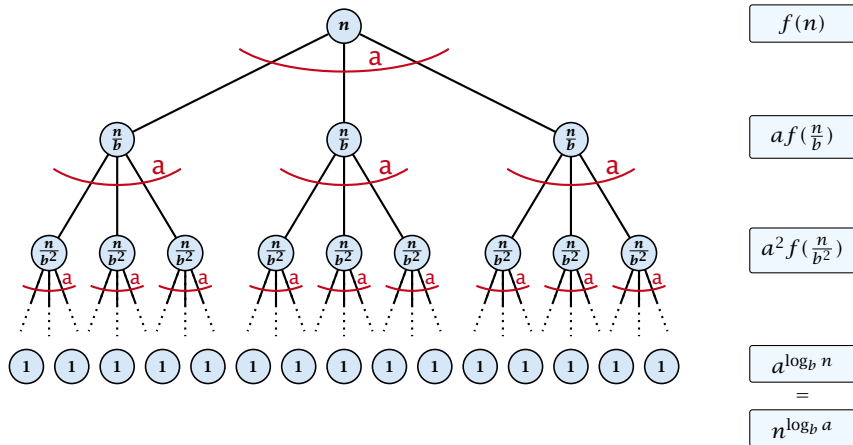
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6.2 Master Theorem

This gives

$$T(n) = n^{\log_b a} + \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right).$$

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq c \sum_{i=0}^{\log_b n - 1} a^i \left(\frac{n}{b^i}\right)^{\log_b a - \epsilon} \end{aligned}$$

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$$\boxed{\sum_{i=0}^k q^i = \frac{q^{k+1} - 1}{q - 1}} = cn^{\log_b a - \epsilon} (b^{\epsilon \log_b n} - 1) / (b^{\epsilon} - 1)$$

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Hence,

$$T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)}$$

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Hence,

$$T(n) \leq \left(\frac{c}{b^{\epsilon} - 1} + 1 \right) n^{\log_b(a)} \quad \Rightarrow T(n) = \mathcal{O}(n^{\log_b a}).$$

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Hence,

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Hence,

$$T(n) = \Omega(n^{\log_b a} \log_b n) \quad \Rightarrow T(n) = \Omega(n^{\log_b a} \log n).$$

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$n = b^\ell \Rightarrow \ell = \log_b n$	$= cn^{\log_b a} \sum_{i=0}^{\ell-1} \left(\log_b \left(\frac{b^\ell}{b^i}\right)\right)^k$
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$$= cn^{\log_b a} \sum_{i=1}^{\ell} i^k \approx \frac{1}{k} \ell^{k+1}$$

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$$\Rightarrow T(n) = \mathcal{O}(n^{\log_b a} \log^{k+1} n).$$

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

Where did we use $f(n) \geq \Omega(n^{\log_b a + \epsilon})$?

Case 3. Now suppose that $f(n) \geq dn^{\log_b a + \epsilon}$, and that for sufficiently large n : $af(n/b) \leq cf(n)$, for $c < 1$.

From this we get $a^i f(n/b^i) \leq c^i f(n)$, where we assume that $n/b^{i-1} \geq n_0$ is still sufficiently large.

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$$T(n) - n^{\log_b a} = \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right)$$

Where did we use $f(n) \geq \Omega(n^{\log_b a + \epsilon})$?

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

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$$q < 1 : \sum_{i=0}^n q^i = \frac{1 - q^{n+1}}{1 - q} \leq \frac{1}{1 - q}$$

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Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

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$$\begin{aligned} T(n) - n^{\log_b a} &= \sum_{i=0}^{\log_b n - 1} a^i f\left(\frac{n}{b^i}\right) \\ &\leq \sum_{i=0}^{\log_b n - 1} c^i f(n) + \mathcal{O}(n^{\log_b a}) \\ &\leq \frac{1}{1-c} f(n) + \mathcal{O}(n^{\log_b a}) \end{aligned}$$

$$q < 1 : \sum_{i=0}^n q^i = \frac{1-q^{n+1}}{1-q} \leq \frac{1}{1-q}$$

Hence,

$$T(n) \leq \mathcal{O}(f(n))$$

$$\Rightarrow T(n) = \Theta(f(n)).$$

Where did we use $f(n) \geq \Omega(n^{\log_b a + \epsilon})$?

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

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<hr/>									

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<hr/>									
								1	
								0	

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$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & & & & 1 & & \\ & & & & & & & & 0 & \end{array}$$

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$$\begin{array}{r} 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ A \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ B \\ \hline 0\ 0 \end{array}$$

The diagram illustrates the addition of two 10-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1 and the bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A horizontal line is drawn under the bits of B. The result of the addition is shown below the line, with the bits 0 and 0 highlighted in a light blue box. The carry bits are indicated by small '1's below the 7th and 8th bits of B.

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For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
						0	1	1	
							0	0	

The diagram illustrates the addition of two 9-bit integers, A and B. A vertical line is drawn between the 7th and 8th bits. A light blue shaded box highlights the 7th bit of A (1) and the 7th bit of B (0). Below the line, the 7th bit of the sum is 0, and the 8th bit of the sum is 1. Below the 8th bit, the 8th bit of the sum is 0, and the 9th bit of the sum is 0.

Example: Multiplying Two Integers

Suppose we want to multiply two n -bit Integers, but our registers can only perform operations on integers of constant size.

For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & & & 0 & 0 & 0 & \end{array}$$

The diagram illustrates the addition of two 9-bit integers, A and B. The bits of A are 1, 1, 0, 1, 1, 0, 1, 0, 1 and the bits of B are 1, 0, 0, 0, 1, 0, 0, 1, 1. A horizontal line is drawn under the bits of B. The result of the addition is shown below the line, with the bits 0, 0, 0 highlighted in a light blue box. The carry bits are indicated by small '1's below the bits of B.

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For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
					1	1	1		
						0	0	0	

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1	0	0	0	1	0	0	1	1	B
					0	1	1	1	
					1	0	0	0	

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For this we first need to be able to add two integers A and B :

$$\begin{array}{rcccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & A \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & B \\ \hline & & & & 0 & 1 & 1 & 1 & & \\ & & & & & 1 & 0 & 0 & 0 & \end{array}$$

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				1	0	1	1	1	
-----				0	1	0	0	0	

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The diagram illustrates the addition of two 9-bit integers, A and B, to produce a 9-bit result. The numbers are aligned to the right. A vertical box highlights the 4th bit position (from the right), which contains a '1' from A and a '0' from B. Below this position, a '1' is written, indicating a carry into the 5th bit position. The result row shows the sum of the bits in each column, with carries indicated by small subscripts below the digits.

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For this we first need to be able to add two integers A and B :

1	1	0	1	1	0	1	0	1	A
1	0	0	0	1	0	0	1	1	B
<hr/>									
			0	0	1	0	0	0	

The diagram illustrates the addition of two integers A and B . The bits of A are 1 1 0 1 1 0 1 0 1 and the bits of B are 1 0 0 0 1 0 0 1 1. A vertical box highlights the third bit from the right (the bit '0' in A and the bit '0' in B), indicating the constant size of the register. Below the bits of B , carry bits are shown as subscripts: 1, 1, 0, 1, 1, 1. The result of the addition is 0 0 1 0 0 0.

Example: Multiplying Two Integers

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For this we first need to be able to add two integers A and B :

		1	1	0	1	1	0	1	0	1	A
		1	0	0	0	1	0	0	1	1	B
			0	1	1	0	1	1	1		
				1	0	0	1	0	0	0	

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	1	1	0	0	1	0	0	0		

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	0	0	1	1	0	1	1	1		
		1	1	0	0	1	0	0	0	

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	1	0	0	1	1	0	1	1	1	
	0	1	1	0	0	1	0	0	0	

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This gives that two n -bit integers can be added in time $\mathcal{O}(n)$.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \\ \times 1\ 0\ 1\ 1 \\ \hline \end{array}$$

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Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \times 1011 \\ \hline \end{array}$$

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$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 00001 \\ 00000 \\ 10001 \\ \hline 0 \end{array}$$

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Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 100010 \\ 0000000 \\ 00000000 \end{array}$$

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Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0 \end{array}$$

- This is also known as the “school method” for multiplying integers.
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Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 100010 \\ 0000000 \\ 00001000 \\ \hline 100011001 \end{array}$$

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline \end{array}$$

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$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

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Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 10001 \times 1011 \\ \hline 10001 \\ 100010 \\ 0000000 \\ 00001000 \\ \hline 10111011 \end{array}$$

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Time requirement:

Example: Multiplying Two Integers

Suppose that we want to multiply an n -bit integer A and an m -bit integer B ($m \leq n$).

$$\begin{array}{r} 1\ 0\ 0\ 0\ 1 \times 1\ 0\ 1\ 1 \\ \hline 1\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ \hline 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1 \end{array}$$

- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Time requirement:

- ▶ Computing intermediate results: $\mathcal{O}(nm)$.

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- This is also known as the “school method” for multiplying integers.
- Note that the intermediate numbers that are generated can have at most $m + n \leq 2n$ bits.

Time requirement:

- ▶ Computing intermediate results: $\mathcal{O}(nm)$.
- ▶ Adding m numbers of length $\leq 2n$: $\mathcal{O}((m+n)m) = \mathcal{O}(nm)$.

Example: Multiplying Two Integers

A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

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Suppose that integers A and B are of length $n = 2^k$, for some k .

$$\boxed{b_{n-1} \quad \dots \quad b_0} \times \boxed{a_{n-1} \quad \dots \quad a_0}$$

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A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .

$$\boxed{b_{n-1} \quad \cdots \quad b_{\frac{n}{2}} \quad b_{\frac{n}{2}-1} \quad \cdots \quad b_0} \times \boxed{a_{n-1} \quad \cdots \quad a_{\frac{n}{2}} \quad a_{\frac{n}{2}-1} \quad \cdots \quad a_0}$$

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Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

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A recursive approach:

Suppose that integers A and B are of length $n = 2^k$, for some k .



Then it holds that

$$A = A_1 \cdot 2^{\frac{n}{2}} + A_0 \text{ and } B = B_1 \cdot 2^{\frac{n}{2}} + B_0$$

Hence,

$$A \cdot B = A_1 B_1 \cdot 2^n + (A_1 B_0 + A_0 B_1) \cdot 2^{\frac{n}{2}} + A_0 B_0$$

Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

```
1: if  $|A| = |B| = 1$  then  
2:   return  $a_0 \cdot b_0$   
3: split  $A$  into  $A_0$  and  $A_1$   
4: split  $B$  into  $B_0$  and  $B_1$   
5:  $Z_2 \leftarrow \text{mult}(A_1, B_1)$   
6:  $Z_1 \leftarrow \text{mult}(A_1, B_0) + \text{mult}(A_0, B_1)$   
7:  $Z_0 \leftarrow \text{mult}(A_0, B_0)$   
8: return  $Z_2 \cdot 2^n + Z_1 \cdot 2^{\frac{n}{2}} + Z_0$ 
```

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```
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4: split  $B$  into  $B_0$  and  $B_1$   
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Example: Multiplying Two Integers

Algorithm 3 $\text{mult}(A, B)$

1: **if** $|A| = |B| = 1$ **then**

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2: **return** $a_0 \cdot b_0$

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3: **split** A into A_0 and A_1

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We get the following recurrence:

$$T(n) = 4T\left(\frac{n}{2}\right) + \mathcal{O}(n) .$$

Example: Multiplying Two Integers

Master Theorem: Recurrence: $T[n] = aT(\frac{n}{b}) + f(n)$.

- ▶ Case 1: $f(n) = \mathcal{O}(n^{\log_b a - \epsilon})$ $T(n) = \Theta(n^{\log_b a})$
- ▶ Case 2: $f(n) = \Theta(n^{\log_b a} \log^k n)$ $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$
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In our case $a = 4$, $b = 2$, and $f(n) = \Theta(n)$. Hence, we are in Case 1, since $n = \mathcal{O}(n^{2-\epsilon}) = \mathcal{O}(n^{\log_b a - \epsilon})$.

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We get a running time of $\mathcal{O}(n^2)$ for our algorithm.

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⇒ Not better than the “school method”.

Example: Multiplying Two Integers

We can use the following identity to compute Z_1 :

A more precise
(correct) analysis
would say that
computing Z_1
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We can use the following identity to compute Z_1 :

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Again we are in Case 1. We get a running time of $\Theta(n^{\log_2 3}) \approx \Theta(n^{1.59})$.

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A huge improvement over the “school method”.