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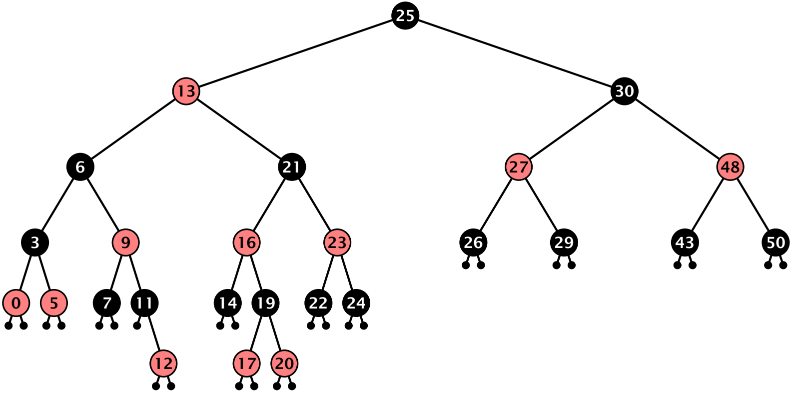
The null-pointers in a binary search tree are replaced by pointers to special null-vertices, that do not carry any object-data

Red Black Trees: Example



7.2 Red Black Trees

Lemma 13

A red-black tree with n internal nodes has height at most $\mathcal{O}(\log n)$.

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The black height $\text{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

7.2 Red Black Trees

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Definition 14
The black height $\mathrm{bh}(v)$ of a node v in a red black tree is the number of black nodes on a path from v to a leaf vertex (not counting v).

We first show:

Lemma 15
A sub-tree of black height $\mathrm{bh}(v)$ in a red black tree contains at least $2^{\mathrm{bh}(v)} - 1$ internal vertices.

7.2 Red Black Trees

Proof of Lemma 15.

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Induction on the height of v.

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- If $\text{height}(v)$ (maximum distance btw. v and a node in the sub-tree rooted at v) is 0 then v is a leaf.

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- If $\text{height}(v)$ (maximum distance btw. v and a node in the sub-tree rooted at v) is 0 then v is a leaf.
- The black height of v is 0.
- The sub-tree rooted at v contains $0 = 2^{\text{bh}(v)} - 1$ inner vertices.

7.2 Red Black Trees

Proof (cont.)

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induction step

- Supose v is a node with $\text{height}(v) > 0$.

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- These children (c_1, c_2) either have $\mathrm{bh}(c_i) = \mathrm{bh}(v)$ or $\mathrm{bh}(c_i) = \mathrm{bh}(v) - 1$.

7.2 Red Black Trees

Proof (cont.)

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- By induction hypothesis both sub-trees contain at least $2^{\mathrm{bh}(v)-1} - 1$ internal vertices.

Proof (cont.)

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- These children (c_1, c_2) either have $\text{bh}(c_i) = \text{bh}(v)$ or $\text{bh}(c_i) = \text{bh}(v) - 1$.
- By induction hypothesis both sub-trees contain at least $2^{\text{bh}(v)-1} - 1$ internal vertices.
- Then T_v contains at least $2(2^{\text{bh}(v)-1} - 1) + 1 \geq 2^{\text{bh}(v)} - 1$ vertices.

□

7.2 Red Black Trees

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The tree contains at least $2^{h/2} - 1$ internal vertices. Hence, $2^{h/2} - 1 \leq n$.

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Hence, the black height of the root is at least $h/2$.

The tree contains at least $2^{h/2} - 1$ internal vertices. Hence, $2^{h/2} - 1 \leq n$.

Hence, $h \leq 2\log(n+1) = \mathcal{O}(\log n)$. $\square$

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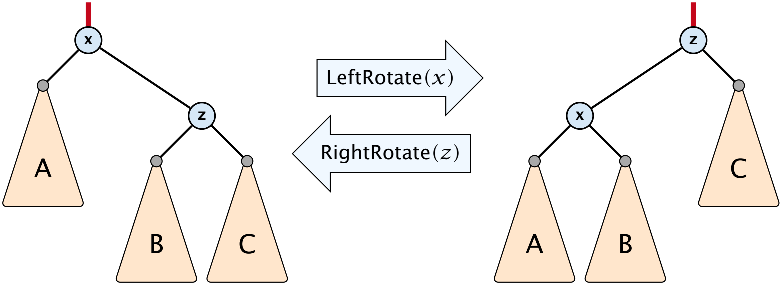
The null-pointers in a binary search tree are replaced by pointers to special null-vertices, that do not carry any object-data.

7.2 Red Black Trees

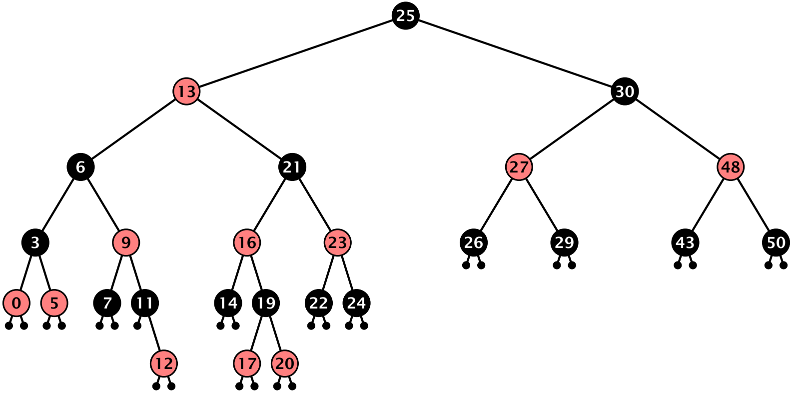
We need to adapt the insert and delete operations so that the red black properties are maintained.

Rotations

The properties will be maintained through rotations:



Red Black Trees: Insert

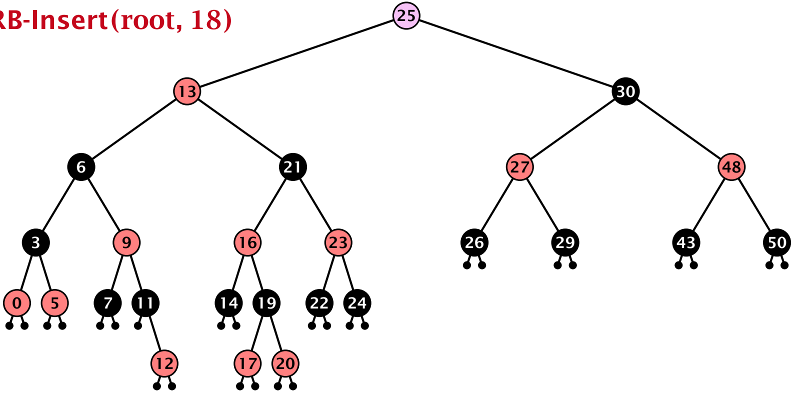


Insert:

- first make a normal insert into a binary search tree
- then fix red-black properties

Red Black Trees: Insert

RB-Insert(root, 18)



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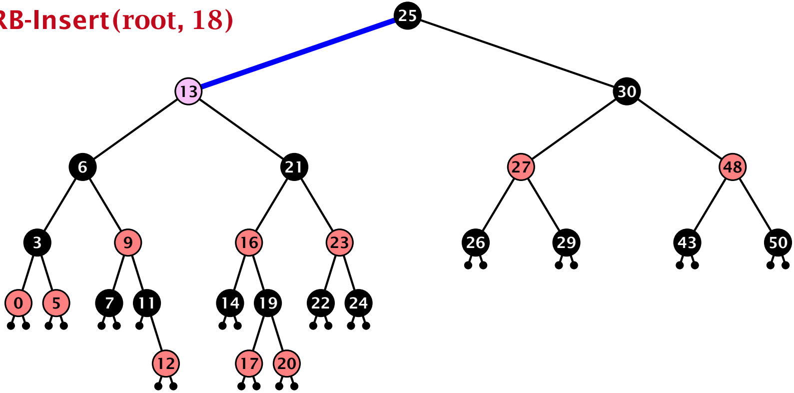

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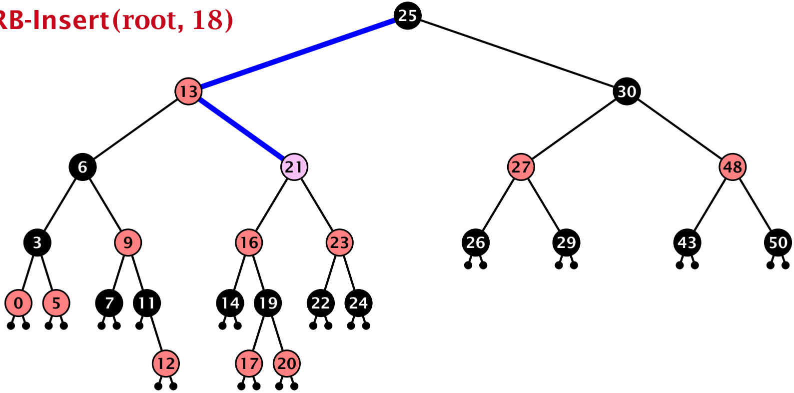

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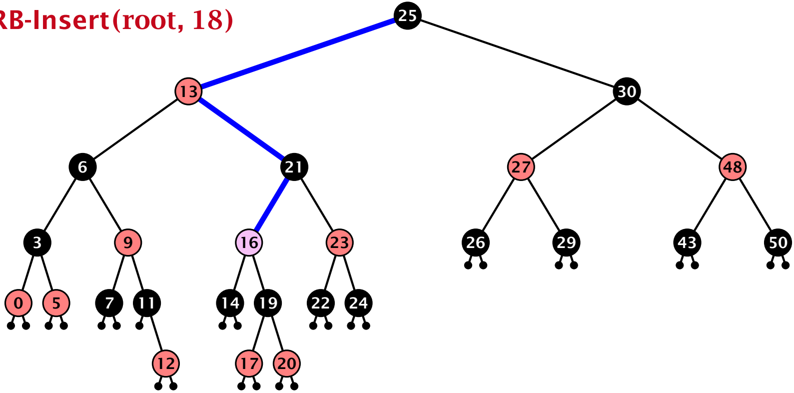

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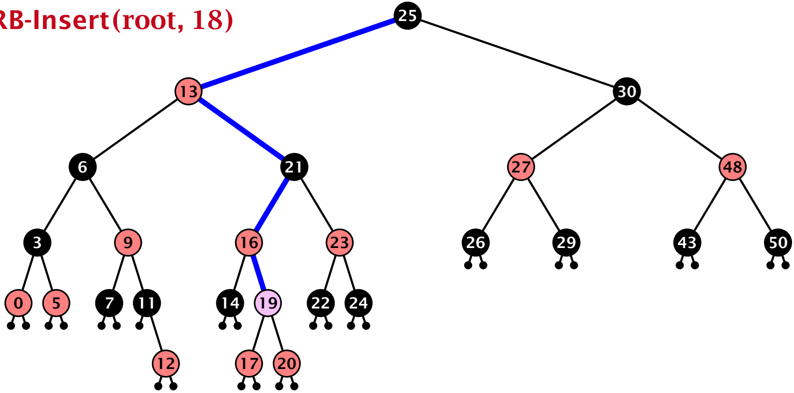

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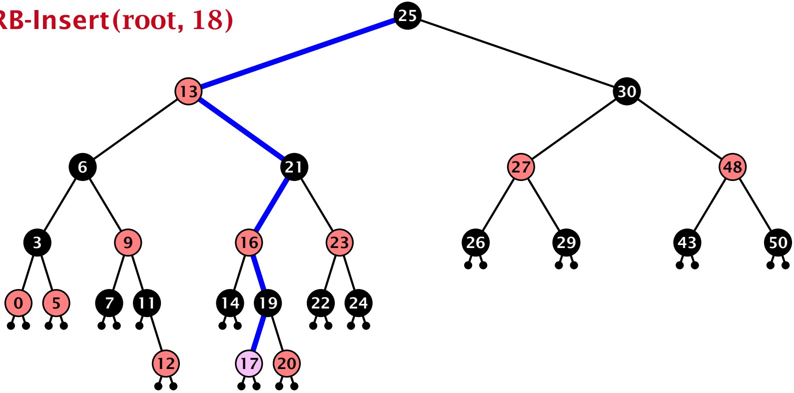

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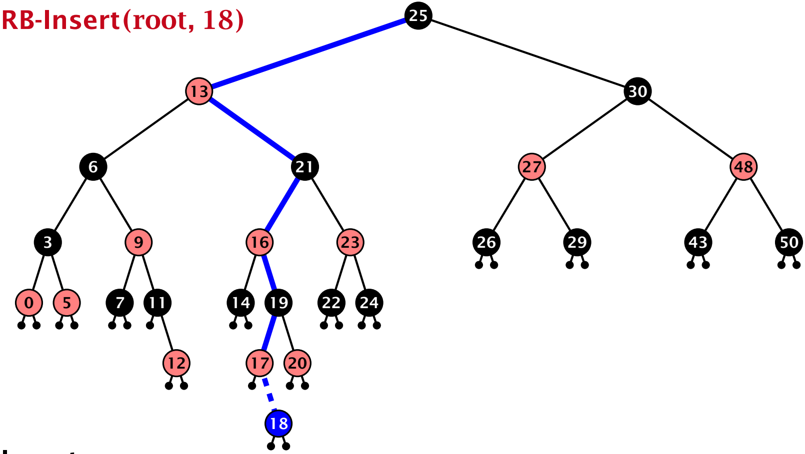

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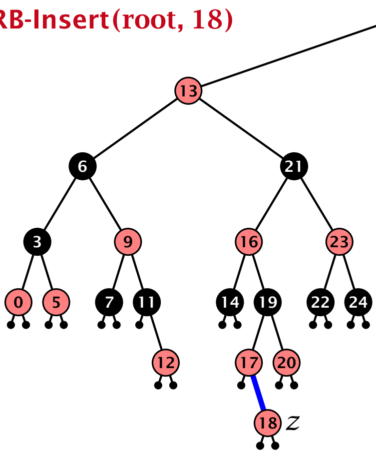

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Invariant of the fix-up algorithm:

- z is a red node

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  - either both of them are red (most important case)
  - or the parent does not exist (violation since root must be black)

If z has a parent but no grand-parent we could simply color the parent/root black; however this case never happens.

Red Black Trees: Insert

Algorithm 10 InsertFix(z)

```
while parent[z] ≠ null and col[parent[z]] = red do
    if parent[z] = left[gp[z]] then
        uncle ← right[grandparent[z]]
        if col[uncle] = red then
            col[p[z]] ← black; col[u] ← black;
            col[gp[z]] ← red; z ← grandparent[z];
        else
            if z = right[parent[z]] then
                z ← p[z]; LeftRotate(z);
            col[p[z]] ← black; col[gp[z]] ← red;
            RightRotate(gp[z]);
    else same as then-clause but right and left exchanged
col(root[T]) ← black;
```

Red Black Trees: Insert

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while parent[z] ≠ null and col[parent[z]] = red do
    if parent[z] = left[gp[z]] then        z in left subtree of grandparent
        uncle ← right[grandparent[z]]
        if col[uncle] = red then
            col[p[z]] ← black; col[u] ← black;
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            col[p[z]] ← black; col[u] ← black;
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            if z = right[parent[z]] then
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            col[gp[z]] ← red; z ← grandparent[z];
        else                                    Case 2: uncle black
            if z = right[parent[z]] then
                z ← p[z]; LeftRotate(z);
            col[p[z]] ← black; col[gp[z]] ← red;
            RightRotate(gp[z]);
    else same as then-clause but right and left exchanged
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Red Black Trees: Insert

Algorithm 10 InsertFix(z)

while parent[z] $\neq$ null **and** col[parent[z]] = red **do**
    if parent[z] = left[gp[z]] **then**
        uncle $\leftarrow$ right[grandparent[z]]
        if col[*uncle*] = red **then**
            col[p[z]] $\leftarrow$ black; col[u] $\leftarrow$ black;
            col[gp[z]] $\leftarrow$ red; $z \leftarrow$ grandparent[z];
        else
            if z = right[parent[z]] **then**   2a: z right child
                $z \leftarrow$ p[z]; LeftRotate(z);
            col[p[z]] $\leftarrow$ black; col[gp[z]] $\leftarrow$ red;
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col(root[T]) $\leftarrow$ black;

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            col[p[z]] ← black; col[gp[z]] ← red;    2b: z left child
            RightRotate(gp[z]);
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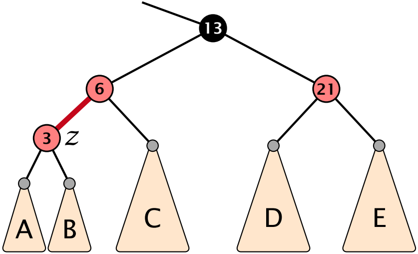
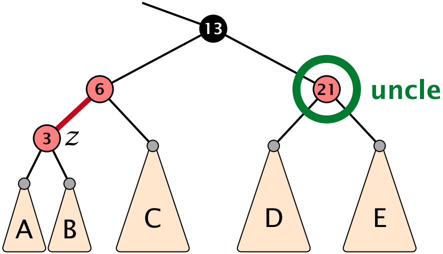
Case 1: Red Uncle



Case 1: Red Uncle



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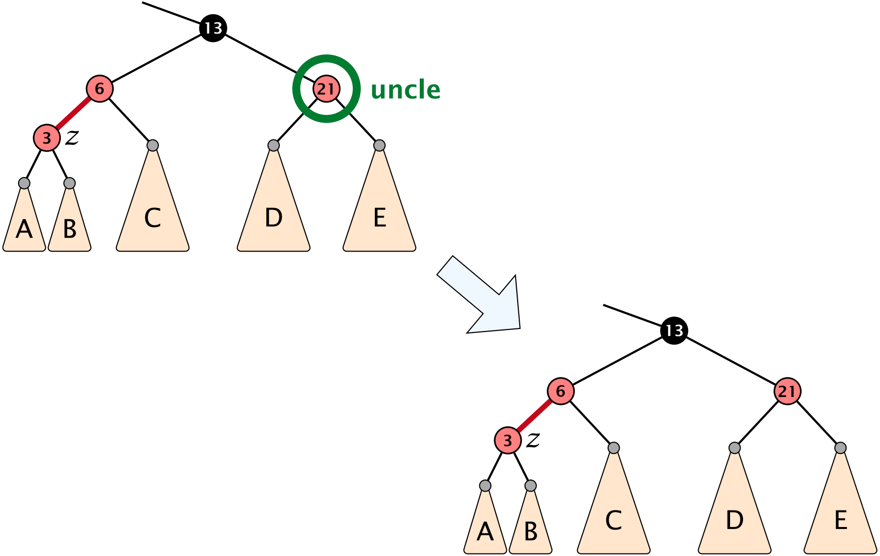

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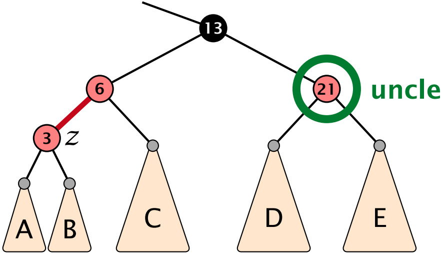
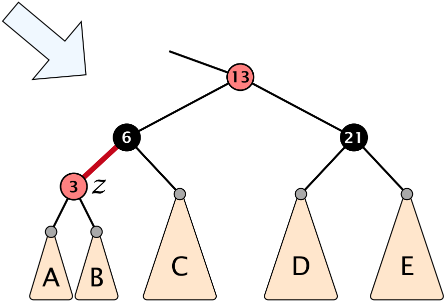
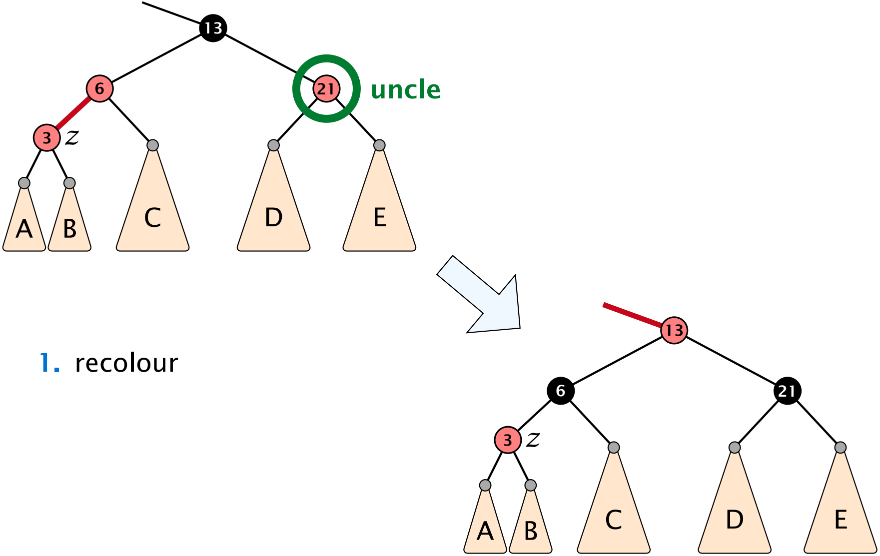




1. recolour

Case 1: Red Uncle



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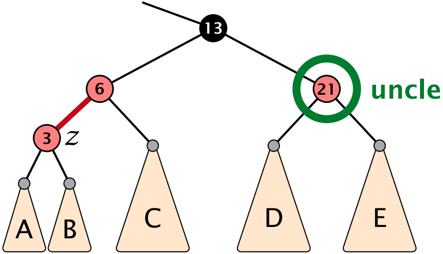
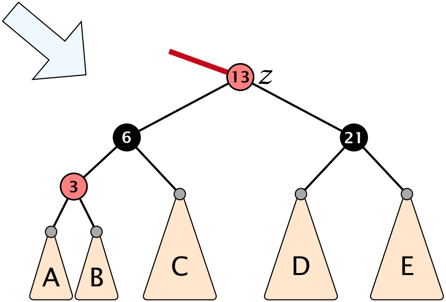


1. recolour
2. move z to grand-parent



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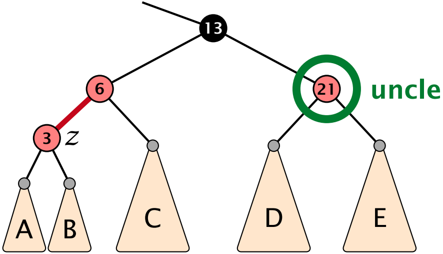
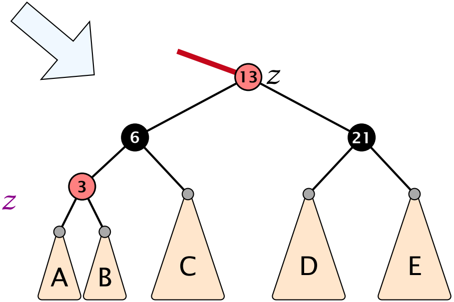
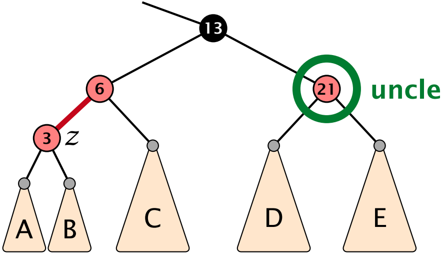
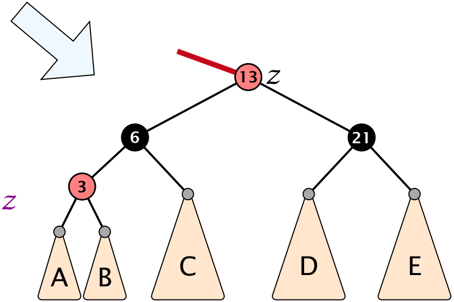


1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z



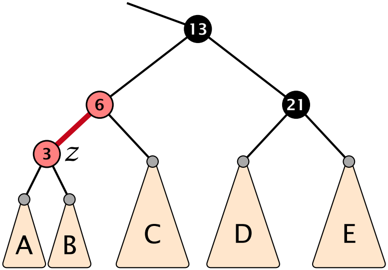
Case 1: Red Uncle



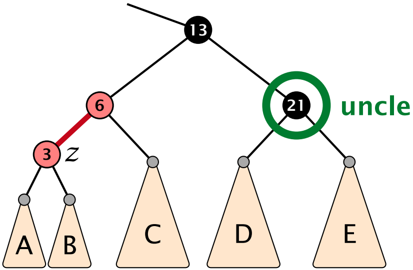


1. recolour
2. move z to grand-parent
3. invariant is fulfilled for new z
4. you made progress

Case 2b: Black uncle and z is left child

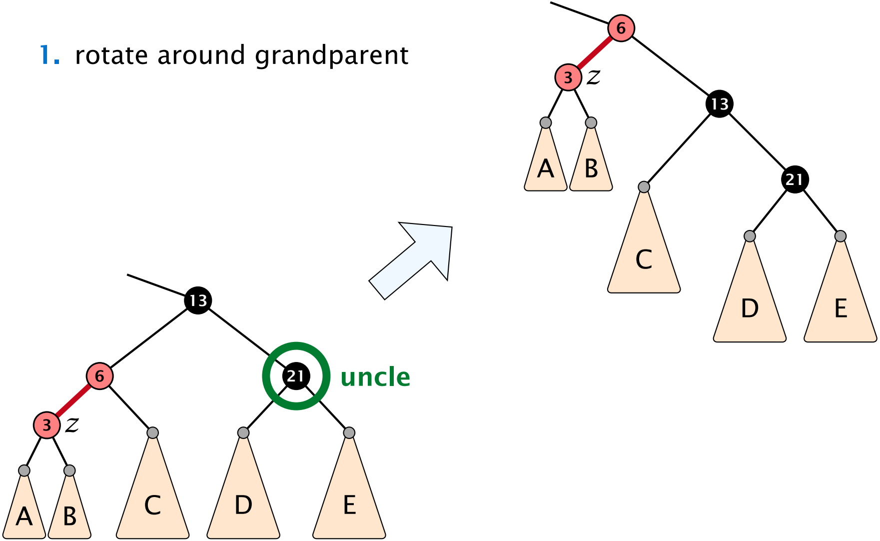


Case 2b: Black uncle and z is left child



Case 2b: Black uncle and z is left child

1. rotate around grandparent



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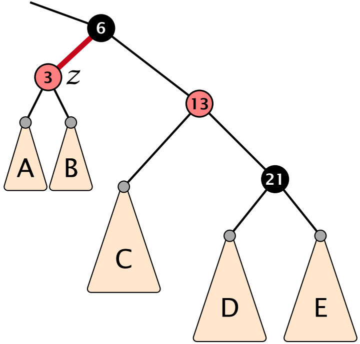
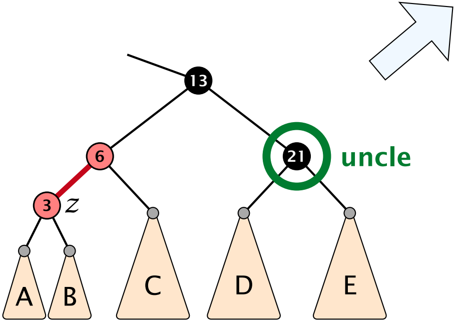
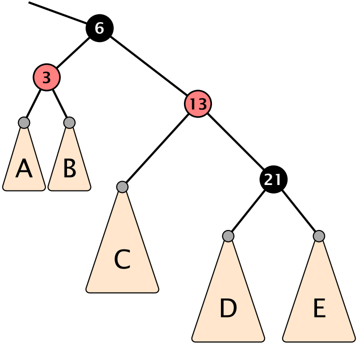


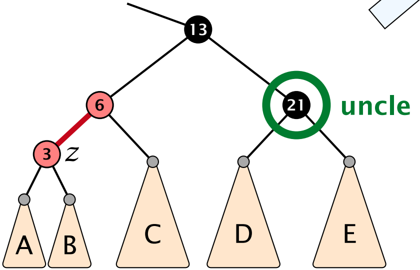
1. rotate around grandparent
2. re-colour to ensure that black height property holds



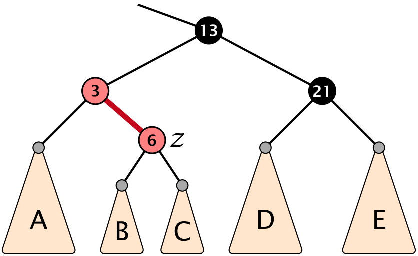
Case 2b: Black uncle and z is left child

1. rotate around grandparent
2. re-colour to ensure that black height property holds
3. you have a red black tree





Case 2a: Black uncle and z is right child



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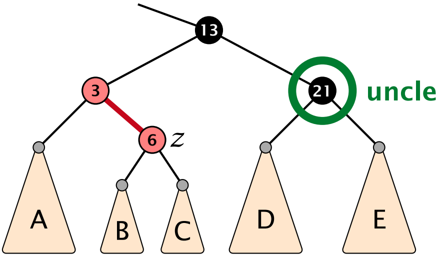
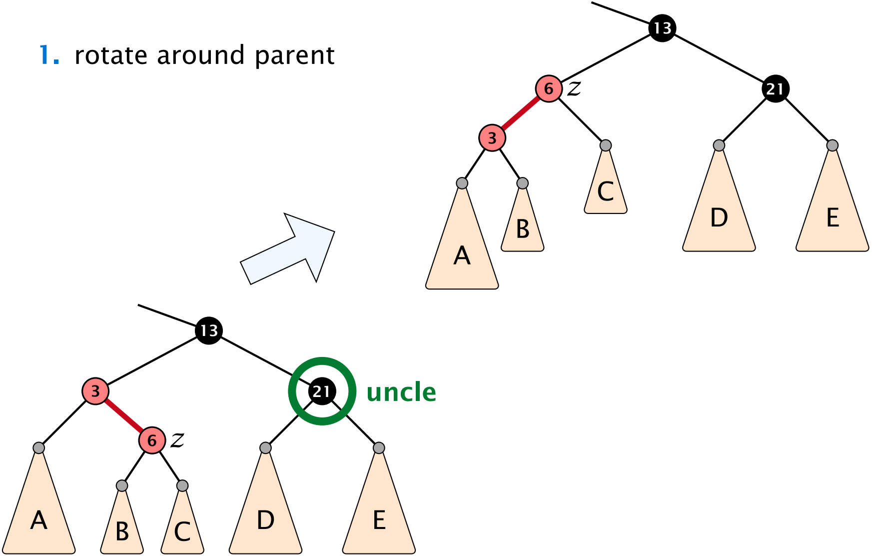


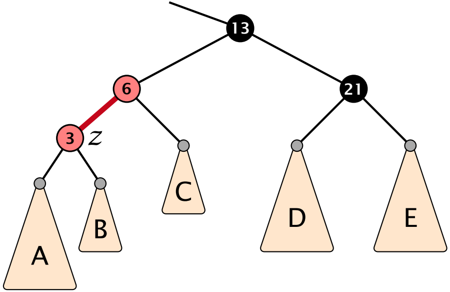
Case 2a: Black uncle and z is right child

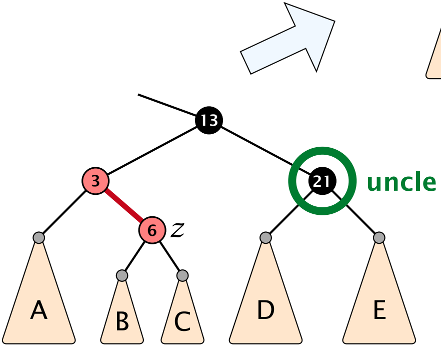
1. rotate around parent



Case 2a: Black uncle and z is right child

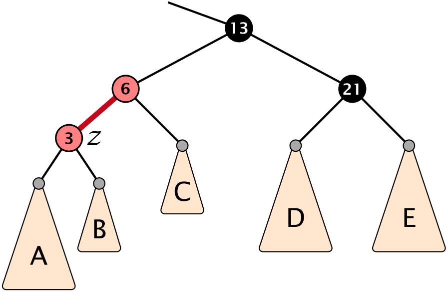
1. rotate around parent
2. move z downwards

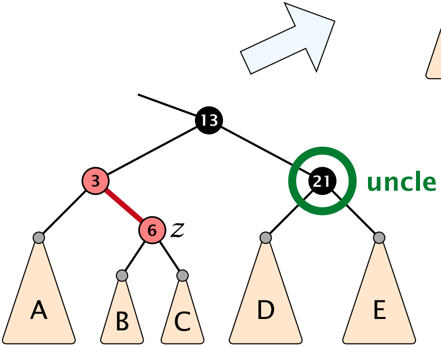




Case 2a: Black uncle and z is right child

1. rotate around parent
2. move z downwards
3. you have Case 2b.





Red Black Trees: Insert

Running time:

- Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.

Red Black Trees: Insert

Running time:

- Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- Case 2a $\rightarrow$ Case 2b $\rightarrow$ red-black tree

Red Black Trees: Insert

Running time:

- Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- Case 2a $\rightarrow$ Case 2b $\rightarrow$ red-black tree
- Case 2b $\rightarrow$ red-black tree

Red Black Trees: Insert

Running time:

- Only Case 1 may repeat; but only $h/2$ many steps, where h is the height of the tree.
- Case 2a $\rightarrow$ Case 2b $\rightarrow$ red-black tree
- Case 2b $\rightarrow$ red-black tree

Performing Case 1 at most $\mathcal{O}(\log n)$ times and every other case at most once, we get a red-black tree. Hence $\mathcal{O}(\log n)$ re-colorings and at most 2 rotations.

Red Black Trees: Delete

Red Black Trees: Delete

First do a standard delete.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

Red Black Trees: Delete

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If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

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If it was black there may be the following problems.

- Parent and child of x were red; two adjacent red vertices.

Red Black Trees: Delete

First do a standard delete.

If the spliced out node x was red everything is fine.

If it was black there may be the following problems.

- Parent and child of x were red; two adjacent red vertices.
- If you delete the root, the root may now be red.

Red Black Trees: Delete

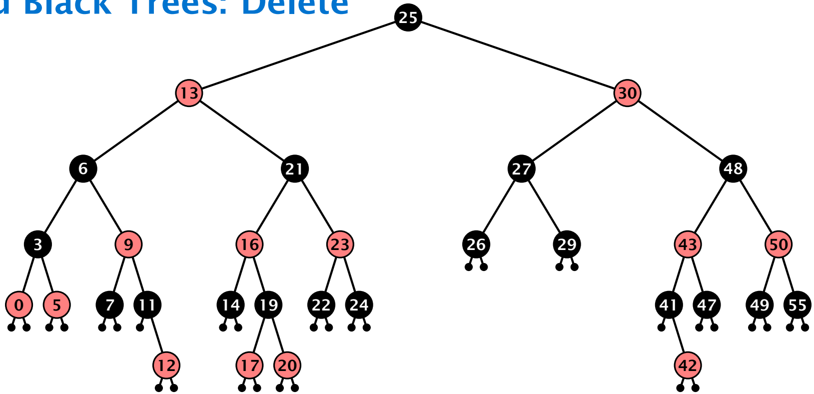
First do a standard delete.

If the spliced out node x was red everything is fine.

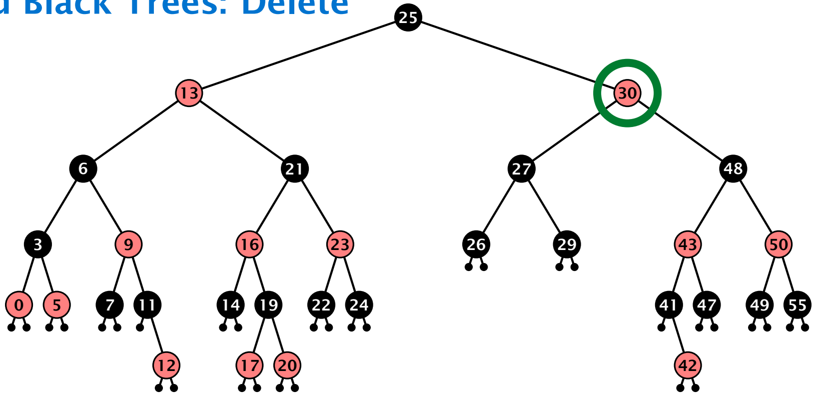
If it was black there may be the following problems.

- Parent and child of x were red; two adjacent red vertices.
- If you delete the root, the root may now be red.
- Every path from an ancestor of x to a descendant leaf of x changes the number of black nodes. Black height property might be violated.

Red Black Trees: Delete



Red Black Trees: Delete

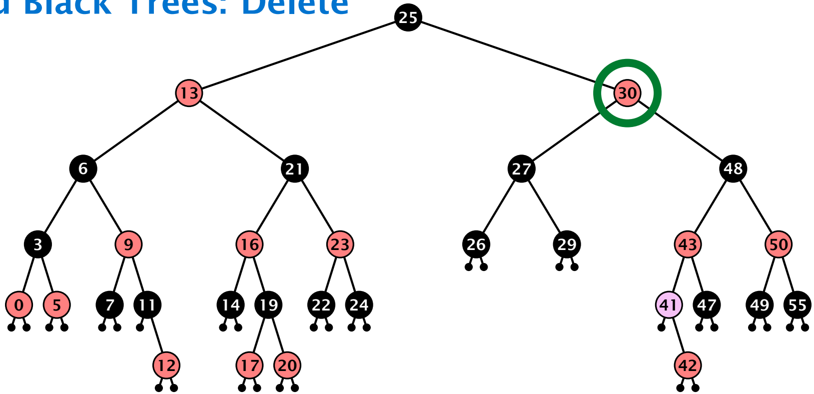


Case 3:

Element has two children

- do normal delete
- when replacing content by content of successor, don't change color of node

Red Black Trees: Delete



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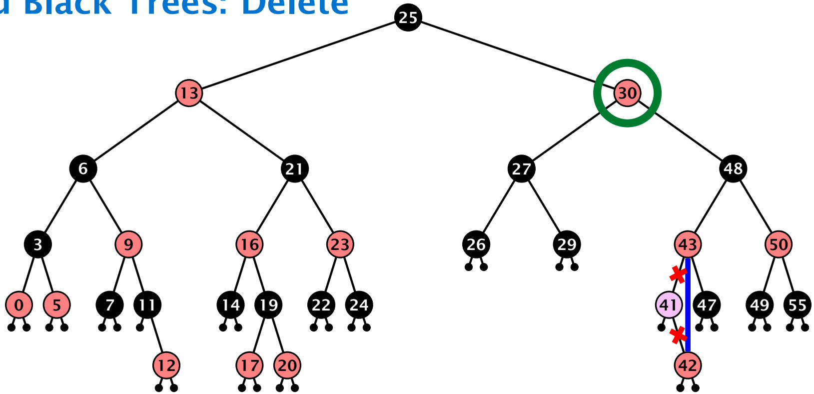

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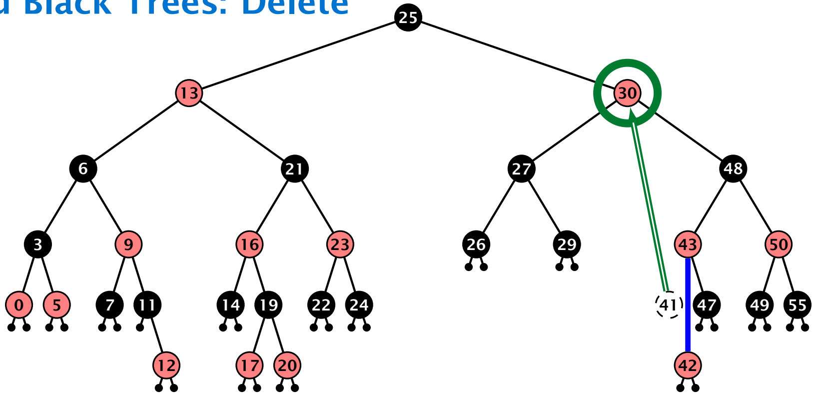
Red Black Trees: Delete



Case 3:

Element has two children

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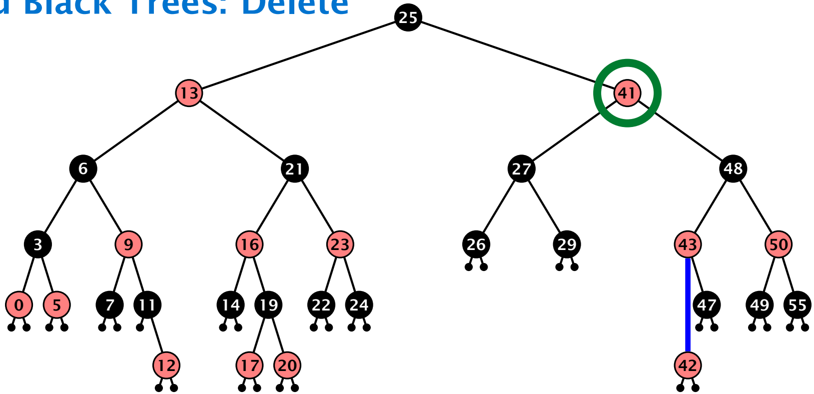
Red Black Trees: Delete

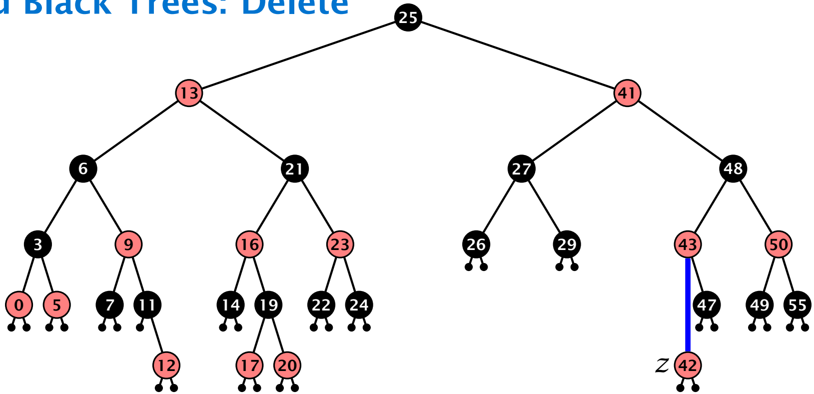


Case 3:

Element has two children

- do normal delete
- when replacing content by content of successor, don't change color of node

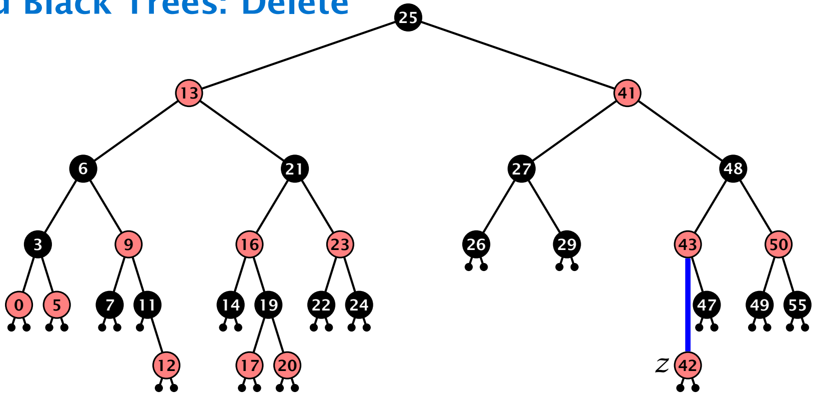
Red Black Trees: Delete



Delete:

- deleting black node messes up black-height property

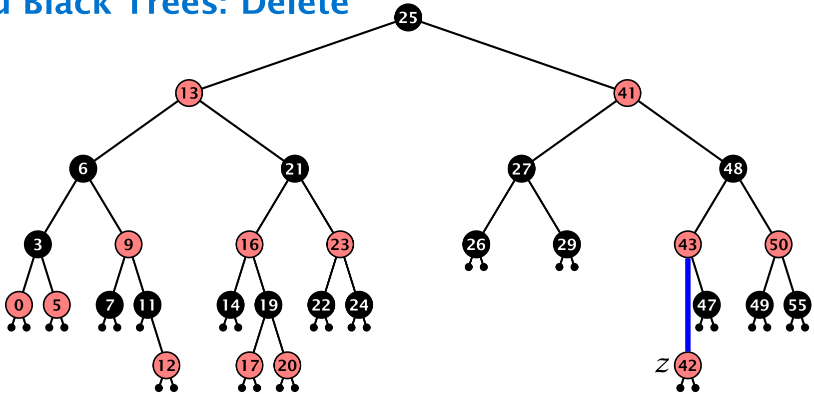
Red Black Trees: Delete



Delete:

- deleting black node messes up black-height property
- if z is red, we can simply color it black and everything is fine

Red Black Trees: Delete



Delete:

- deleting black node messes up black-height property
- if z is red, we can simply color it black and everything is fine
- the problem is if z is black (e.g. a dummy-leaf); we call a fix-up procedure to fix the problem.

Red Black Trees: Delete

Invariant of the fix-up algorithm

- the node z is black

Red Black Trees: Delete

Invariant of the fix-up algorithm

- the node z is black
- if we “assign” a fake black unit to the edge from z to its parent then the black-height property is fulfilled

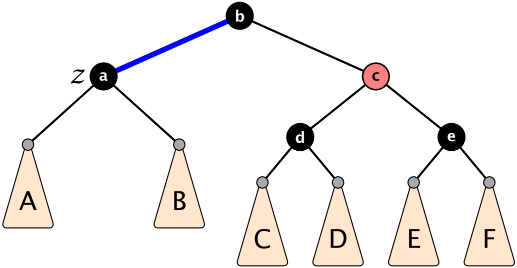
Red Black Trees: Delete

Invariant of the fix-up algorithm

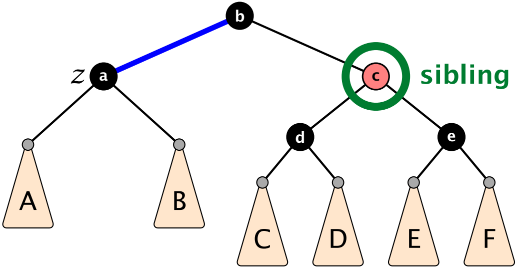
- the node z is black
- if we "assign" a fake black unit to the edge from z to its parent then the black-height property is fulfilled

Goal: make rotations in such a way that you at some point can remove the fake black unit from the edge.

Case 1: Sibling of z is red



Case 1: Sibling of z is red



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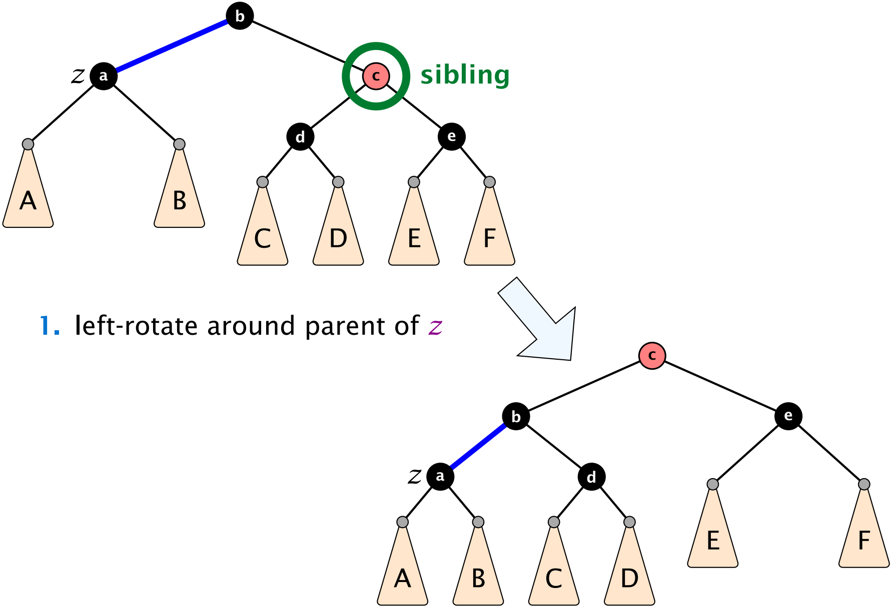
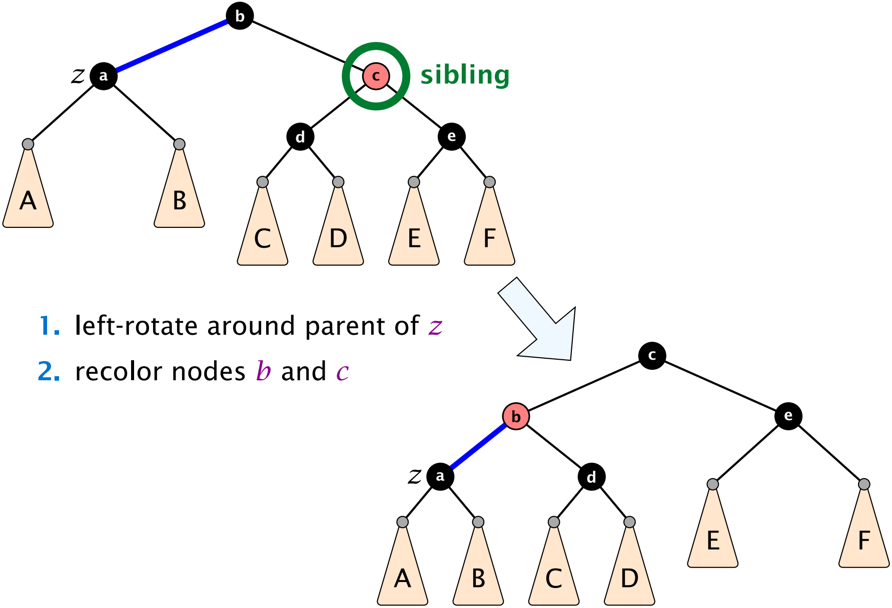


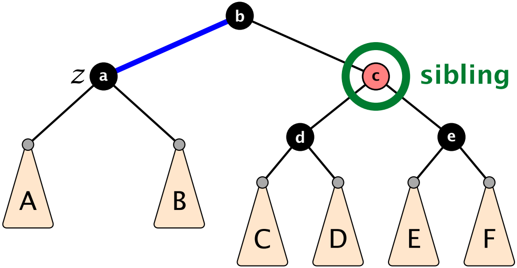
1. left-rotate around parent of z

Case 1: Sibling of z is red

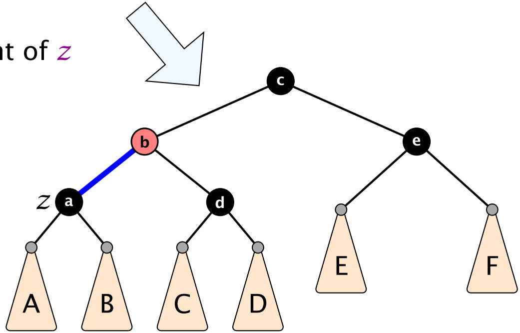


1. left-rotate around parent of z
2. recolor nodes b and c

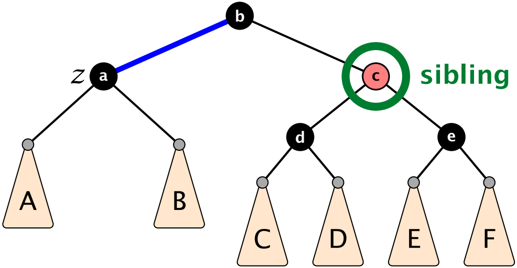
Case 1: Sibling of z is red



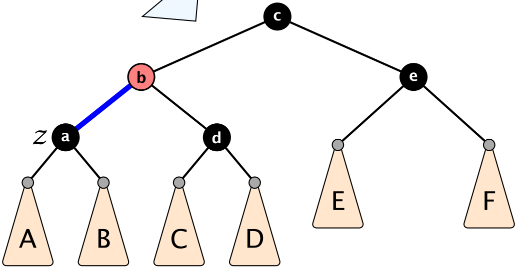
1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black
   (and parent of z is red)



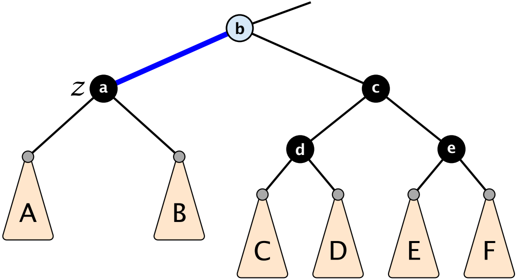
Case 1: Sibling of z is red



1. left-rotate around parent of z
2. recolor nodes b and c
3. the new sibling is black (and parent of z is red)
4. Case 2 (special), or Case 3, or Case 4



Case 2: Sibling is black with two black children



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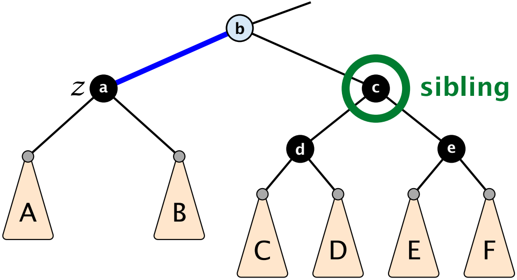

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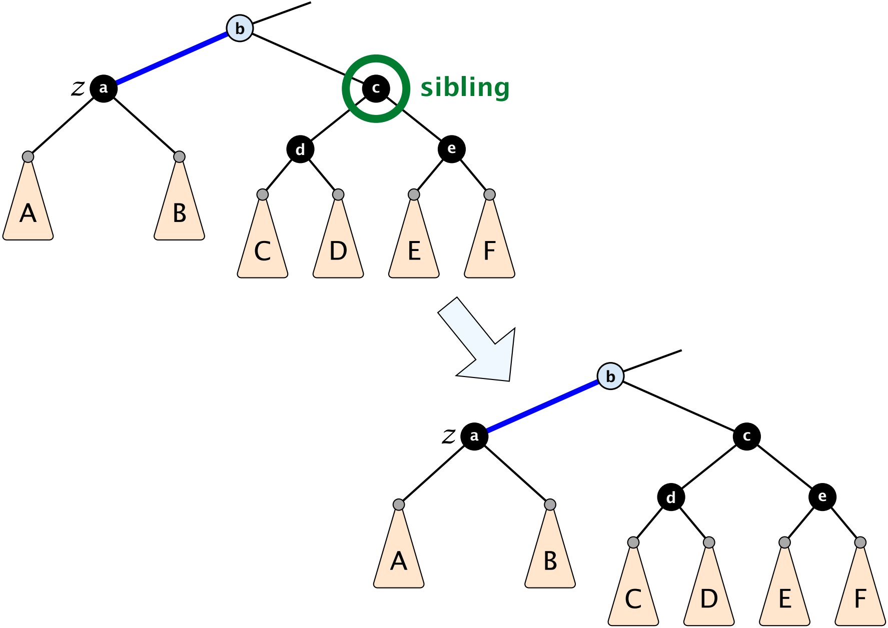

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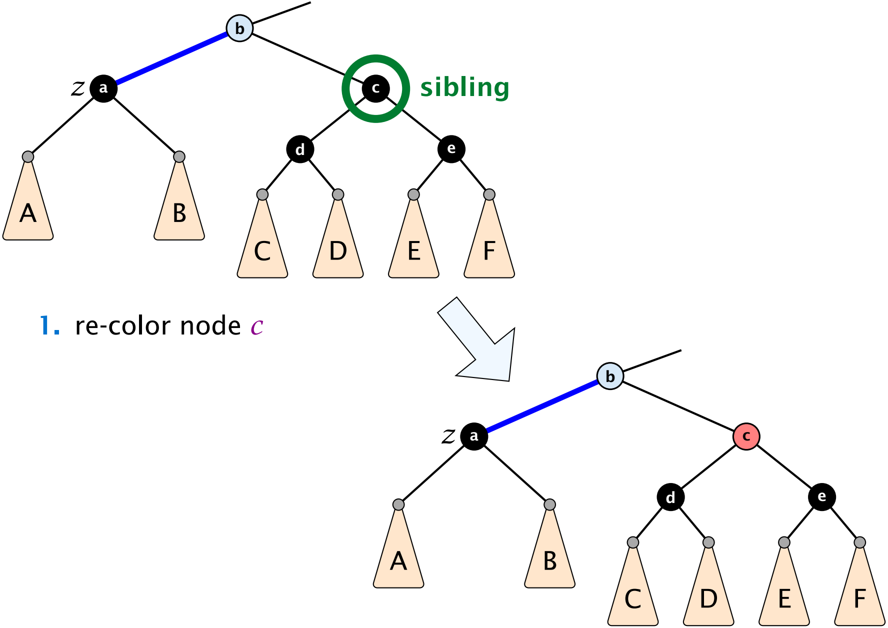
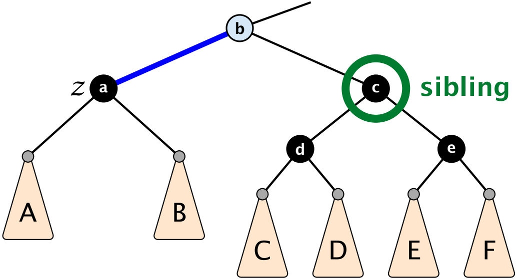
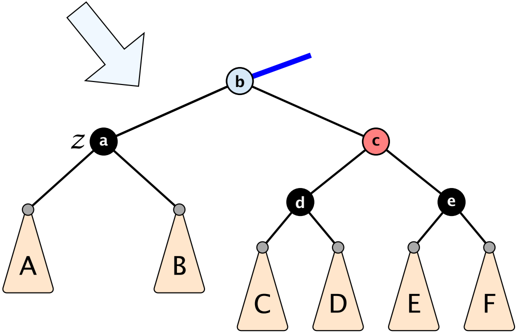


1. re-color node c

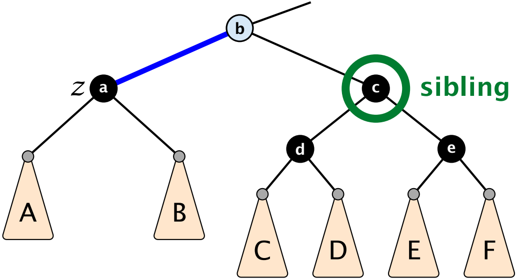
Case 2: Sibling is black with two black children



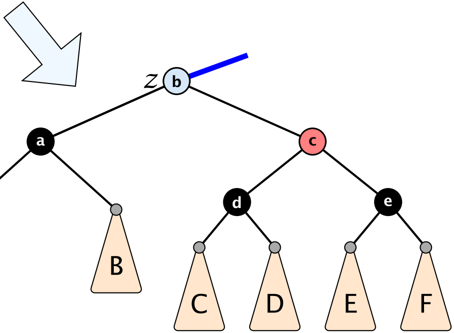
1. re-color node c
2. move fake black unit upwards



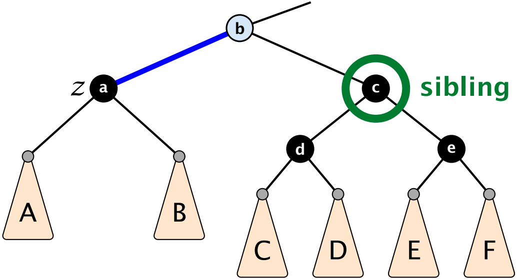
Case 2: Sibling is black with two black children



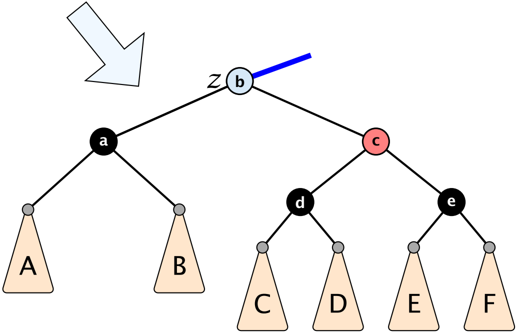
1. re-color node c
2. move fake black unit upwards
3. move z upwards



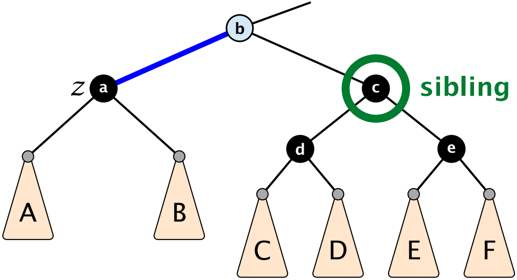
Case 2: Sibling is black with two black children



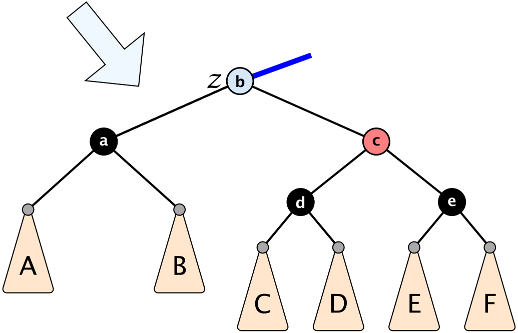
1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress



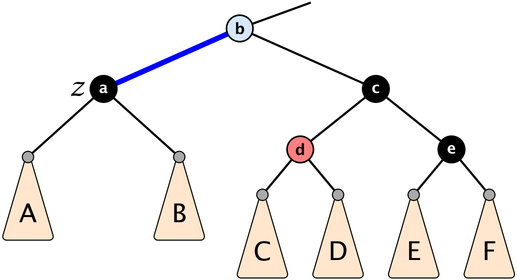
Case 2: Sibling is black with two black children



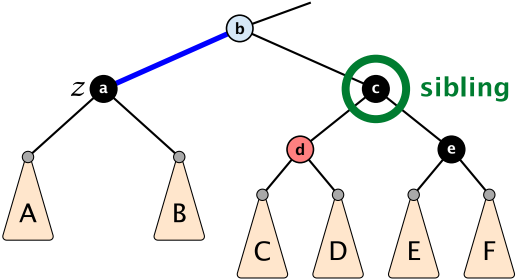
1. re-color node c
2. move fake black unit upwards
3. move z upwards
4. we made progress
5. if b is red we color it black and are done



Case 3: Sibling black with one black child to the right

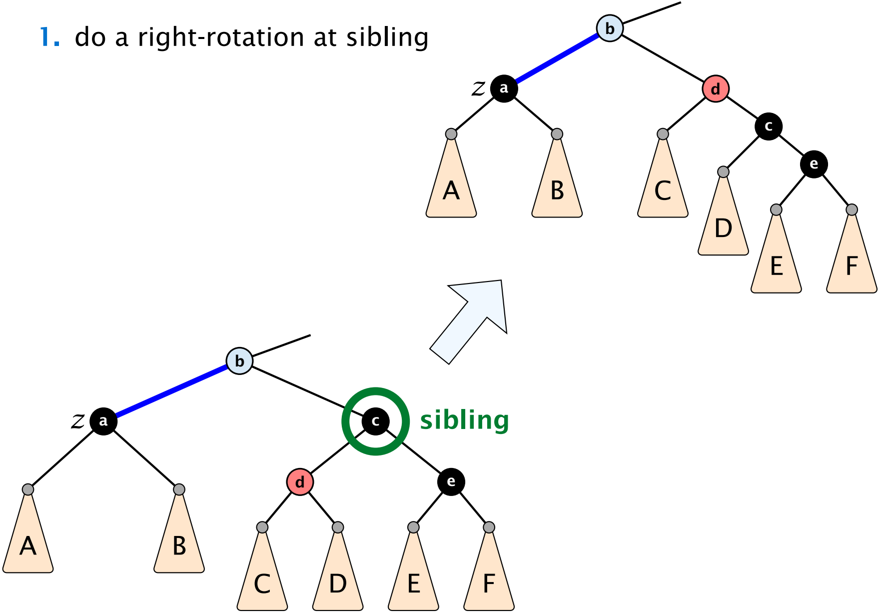


Case 3: Sibling black with one black child to the right



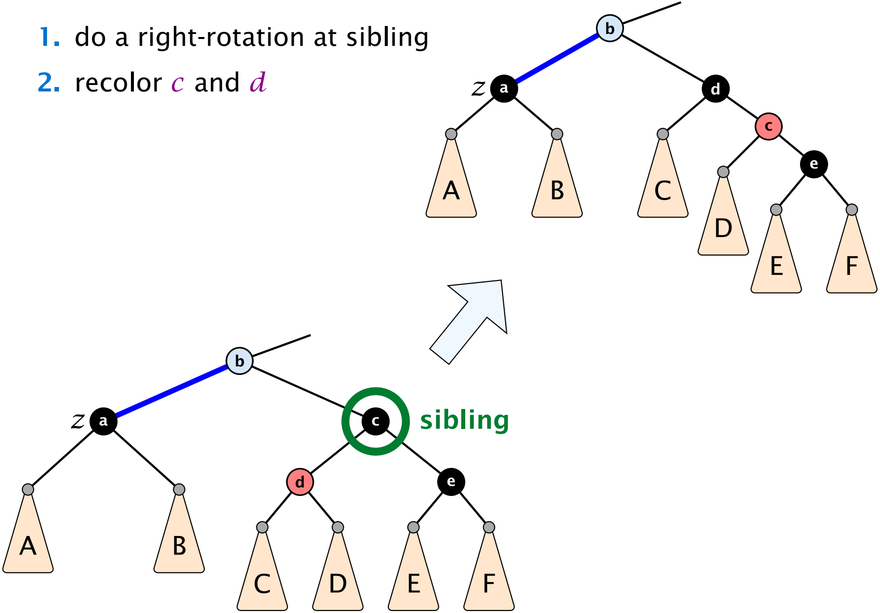
Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling



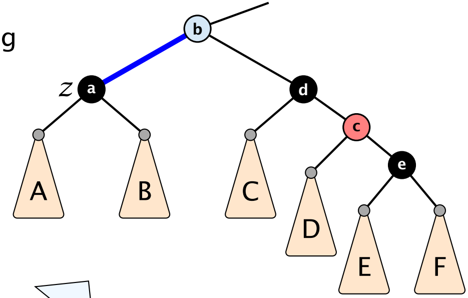
Case 3: Sibling black with one black child to the right

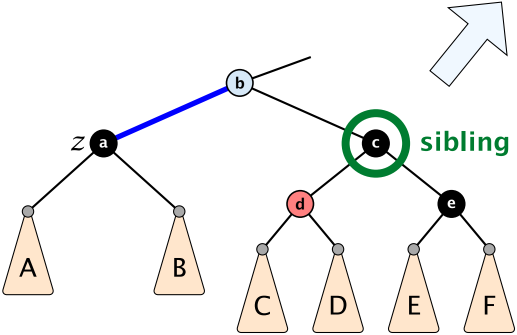
1. do a right-rotation at sibling
2. recolor c and d



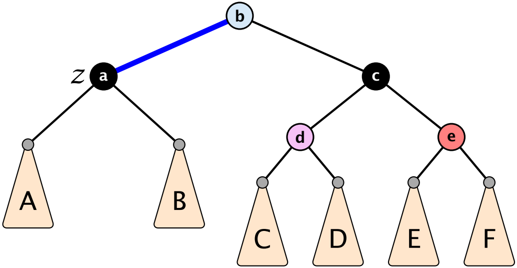
Case 3: Sibling black with one black child to the right

1. do a right-rotation at sibling
2. recolor c and d
3. new sibling is black with red right child (Case 4)

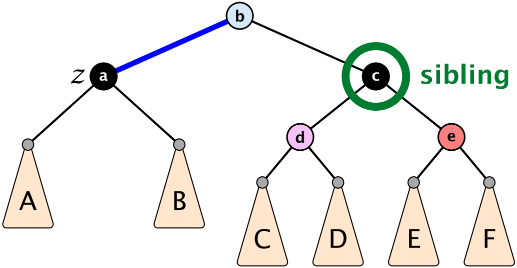




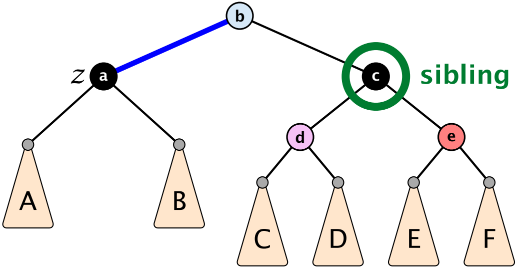
Case 4: Sibling is black with red right child



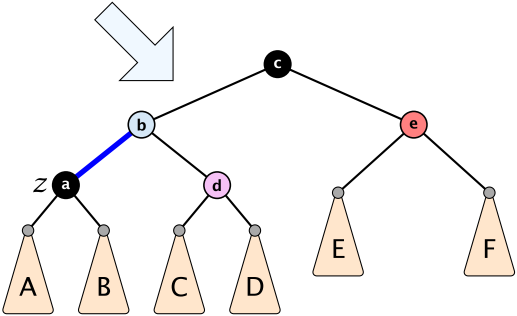
Case 4: Sibling is black with red right child



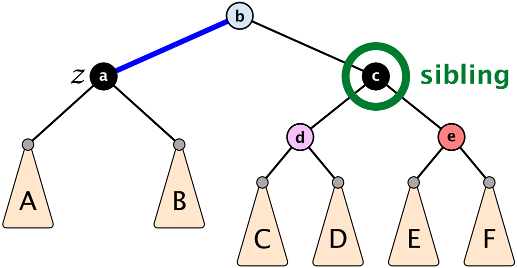
Case 4: Sibling is black with red right child



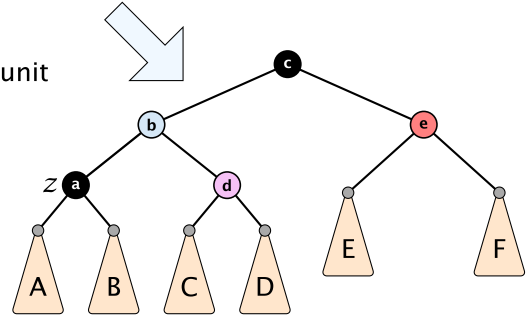
1. left-rotate around b



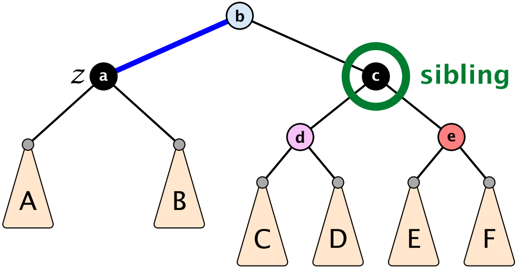
Case 4: Sibling is black with red right child



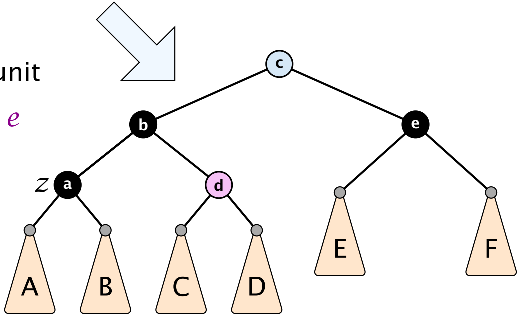
1. left-rotate around b
2. remove the fake black unit



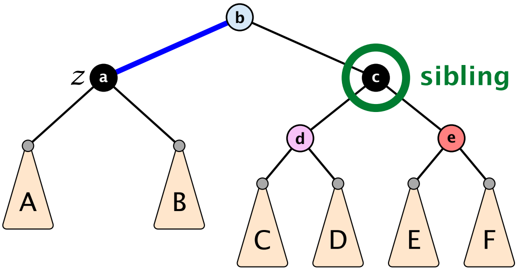
Case 4: Sibling is black with red right child



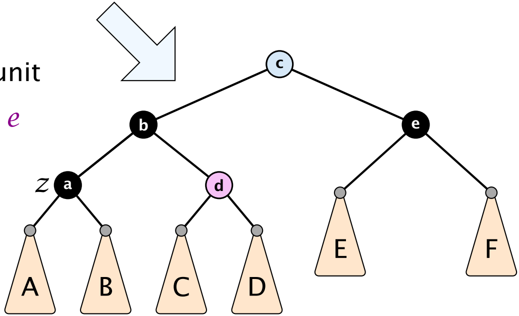
1. left-rotate around b
2. remove the fake black unit
3. recolor nodes b, c, and e



Case 4: Sibling is black with red right child



1. left-rotate around b
2. remove the fake black unit
3. recolor nodes b, c, and e
4. you have a valid red black tree



Running time:

- only Case 2 can repeat; but only h many steps, where h is the height of the tree

Running time:

- only Case 2 can repeat; but only h many steps, where h is the height of the tree
- Case 1 → Case 2 (special) → red black tree
  Case 1 → Case 3 → Case 4 → red black tree
  Case 1 → Case 4 → red black tree

Running time:

- only Case 2 can repeat; but only h many steps, where h is the height of the tree
- Case 1 → Case 2 (special) → red black tree
  Case 1 → Case 3 → Case 4 → red black tree
  Case 1 → Case 4 → red black tree
- Case 3 → Case 4 → red black tree

Running time:

- only Case 2 can repeat; but only h many steps, where h is the height of the tree
- Case 1 → Case 2 (special) → red black tree
  Case 1 → Case 3 → Case 4 → red black tree
  Case 1 → Case 4 → red black tree
- Case 3 → Case 4 → red black tree
- Case 4 → red black tree

Running time:

- only Case 2 can repeat; but only h many steps, where h is the height of the tree
- Case 1 → Case 2 (special) → red black tree
  Case 1 → Case 3 → Case 4 → red black tree
  Case 1 → Case 4 → red black tree
- Case 3 → Case 4 → red black tree
- Case 4 → red black tree

Performing Case 2 at most $\mathcal{O}(\log n)$ times and every other step at most once, we get a red black tree. Hence, $\mathcal{O}(\log n)$ re-colorings and at most 3 rotations.