On Weighted Bipartite Edge Coloring

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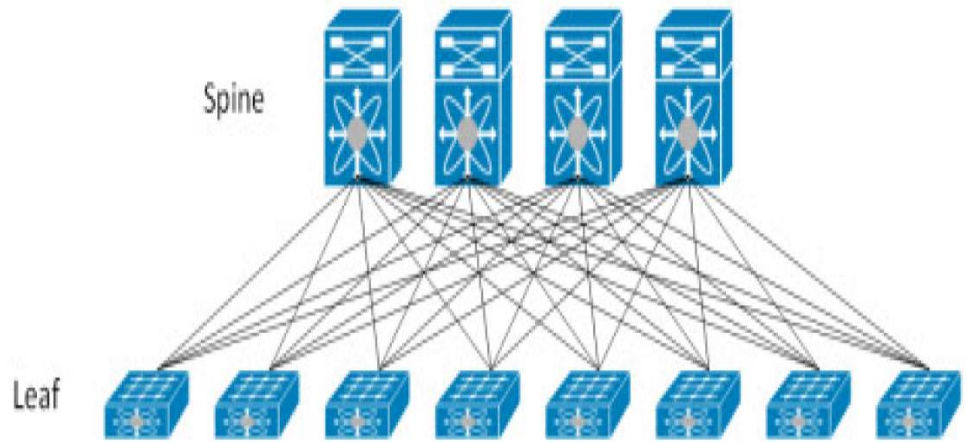
Clos Networks

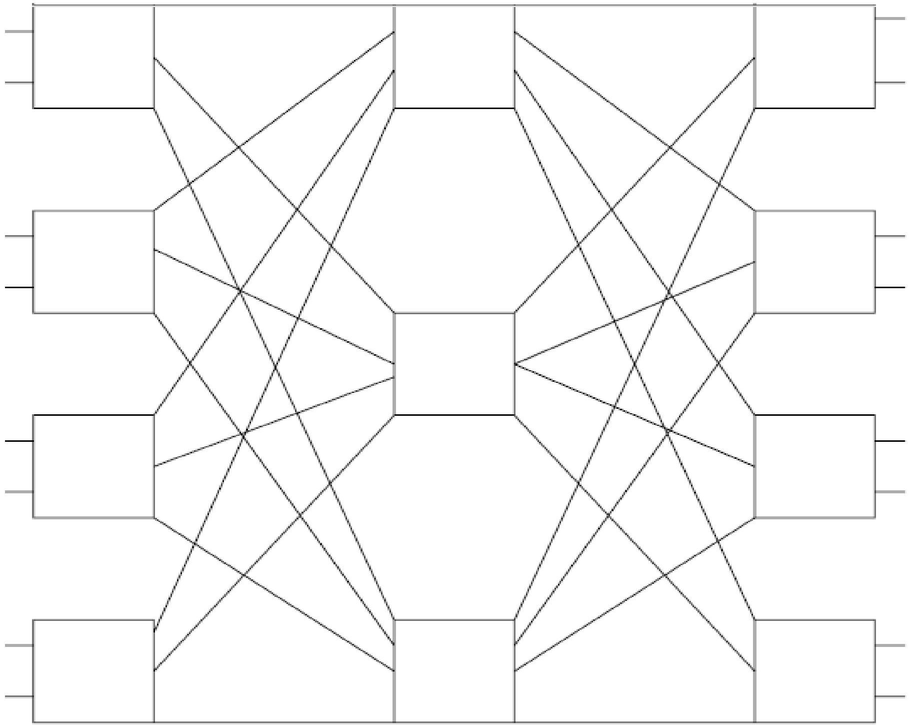
- Clos[1953]: Interconnection networks with small number of links to route simultaneous connection requests such as telephone calls.
- 1990s: Ethernet connectivity.
- 2010s: modern data center networking architectures to achieve high performance and resiliency. [Liu et al. NSDI'13, Akella et al. ICDCN'15, Jyothi et al. SOSR'15, Valadarsky et al. Hotnets'15 etc.]



- Used in layer-2 data center protocol Transparent Interconnect of Lots of Links (TRILL). FabricPath (Cisco), QFabricSystem (Juniper), VCS Fabric (Brocade).

Practical Motivation: Clos Networks

- Clos networks –
  design of interconnection networks with small number of links to route simultaneous connection requests such as telephone calls.



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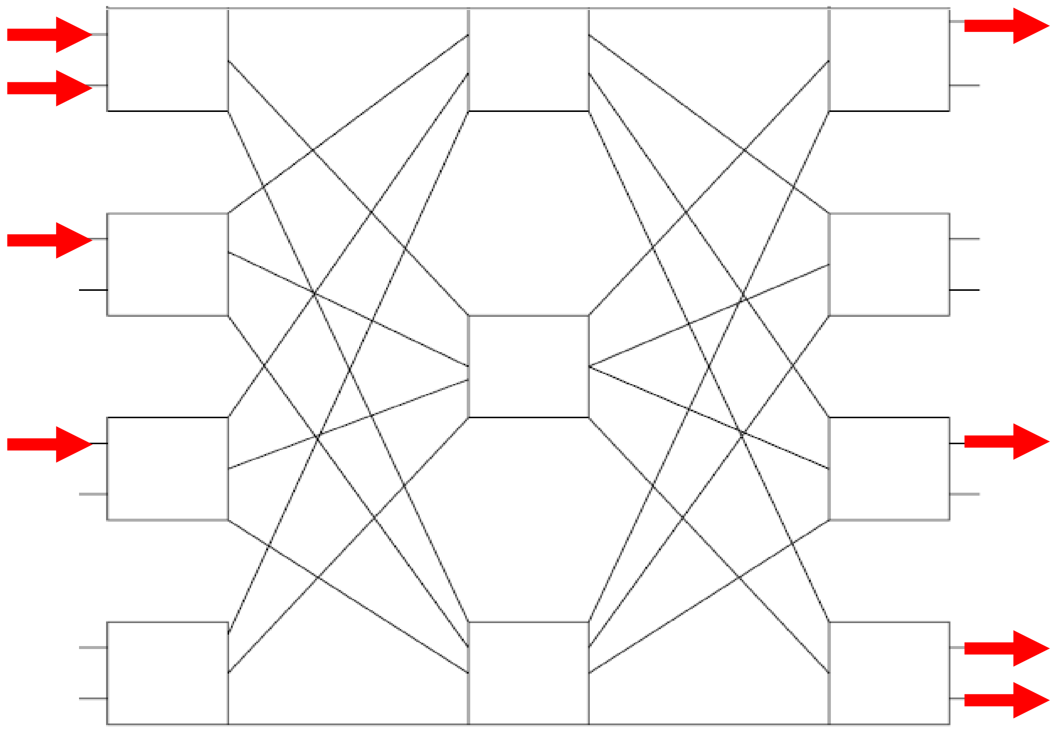
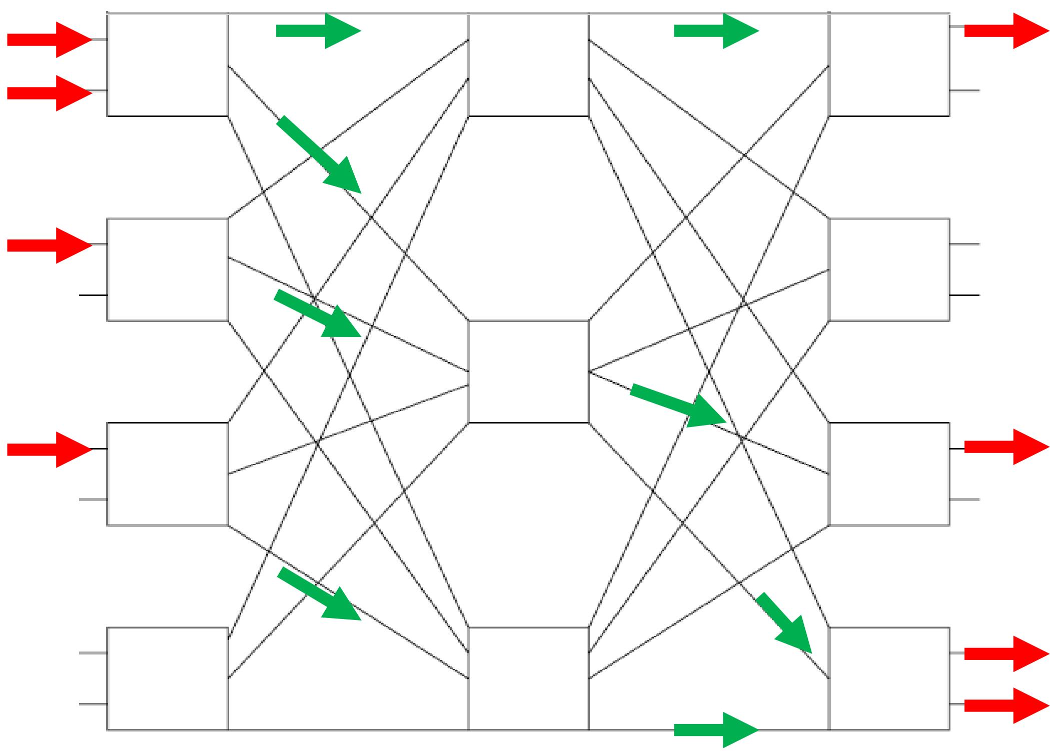


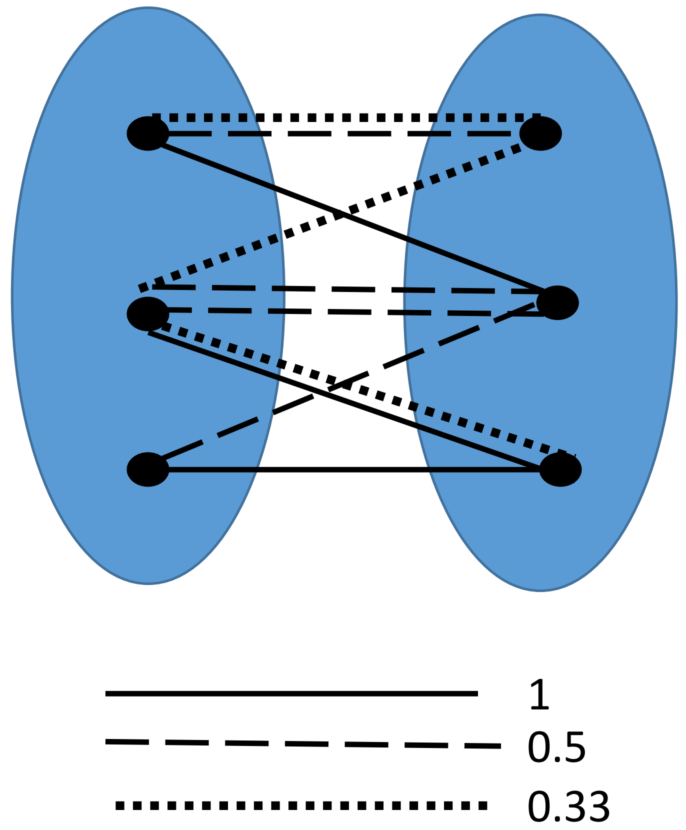
Practical Motivation: Clos Networks

- Clos networks –
  design of interconnection networks with small number of links to route simultaneous connection requests such as telephone calls.
- Rearrangeably nonblocking in the multirate setting: multiple paths for the call to be switched through the network so that calls will always be connected and not "blocked" by another call.
- Minimize number of crossbars in the middle stage.



Weighted Bipartite Edge Coloring

- *Given:* An edge-weighted bipartite multi-graph $G := (V, E)$with edge-weights $w\colon\ E \to [0,1]$.
- $Goal$: Find a proper weighted coloring with minimum number of colors.
- *Proper weighted coloring:* Sum of the edges incident to any vertex of any color is ≤ 1.



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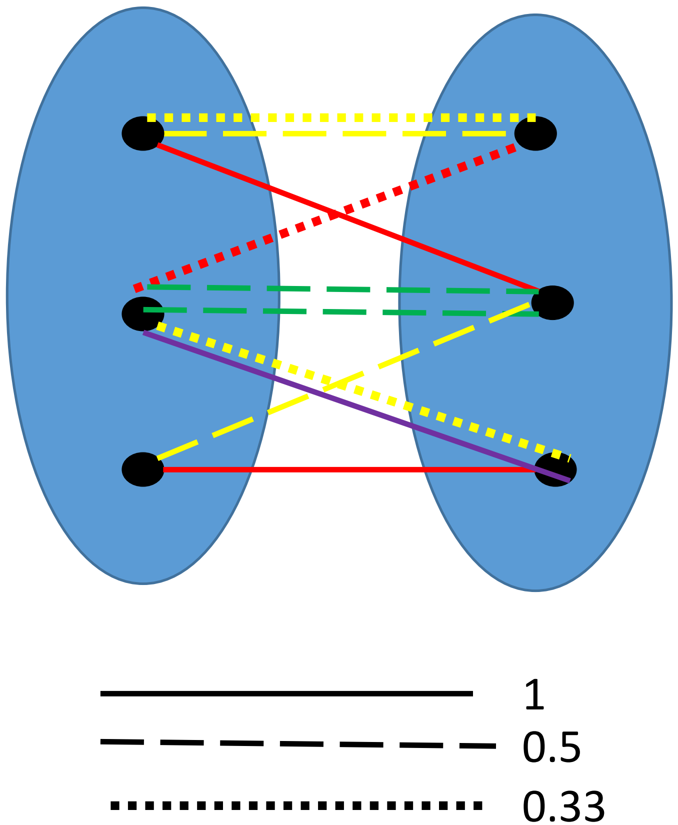
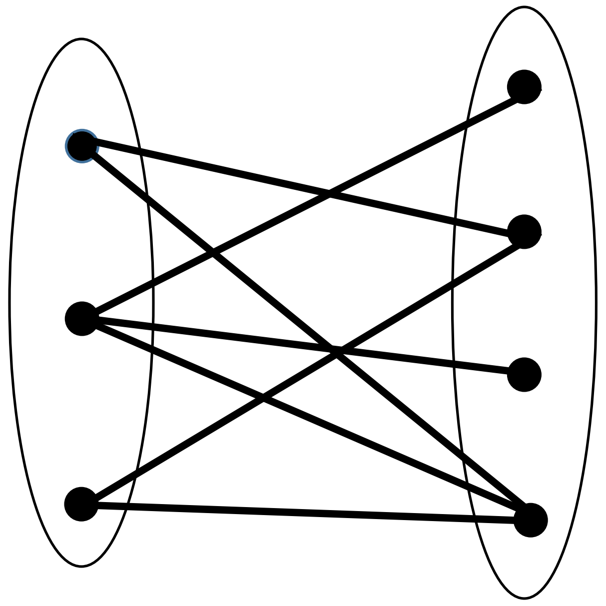


For Theory CS People

Edge Coloring Meets Bin Packing

Bipartite Edge Coloring

- A special case of WBEC when all edge weights are one.
- Chromatic Index $\chi'(G)$: min # of colors required for a proper edge coloring.
- Konig's Theorem:
  For *bipartite graphs* $\chi'(G) = \Delta$.
  where Δ is maximum degree.



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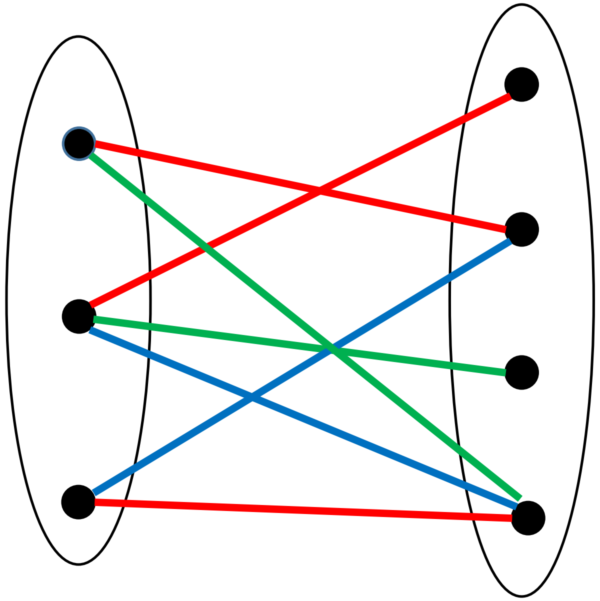
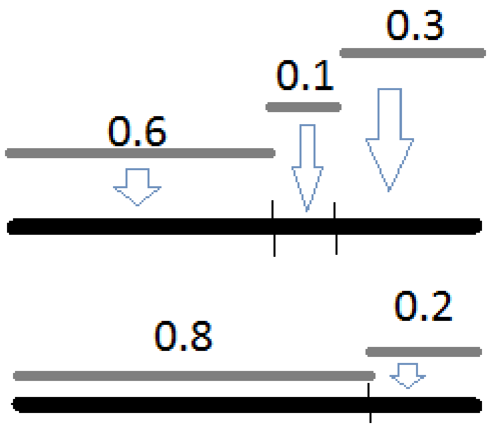
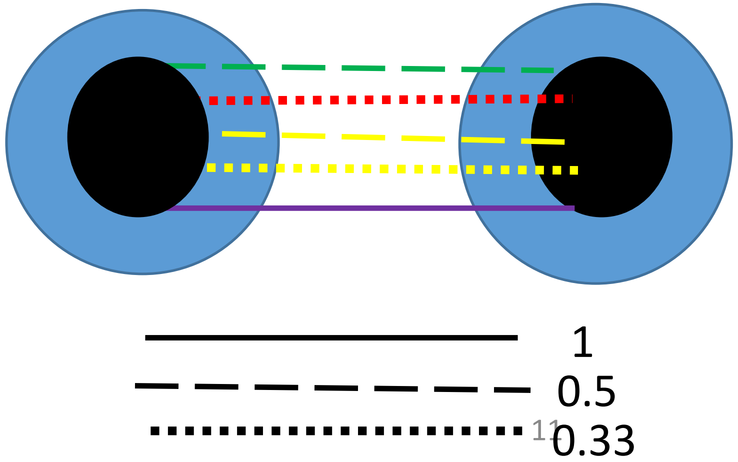


Bin Packing Problem:

- Given : n items with sizes $s_1, s_2 \; \dots s_n$, s.t. $s_i \in (0,1]$
- Goal: Pack all items into min # of unit bins.
- Example: items {0.8, 0.6, 0.3, 0.2, 0.1} can be packed in 2 unit bins: {0.8, 0.2} and {0.6, 0.3, 0.1}.
- NP Hardness from *Partition*
- Approx: $OPT + \log OPT$ [Hoberg-Rothvoss '15]
- Special case of WBEC: $|V_1| = |V_2| = 1$

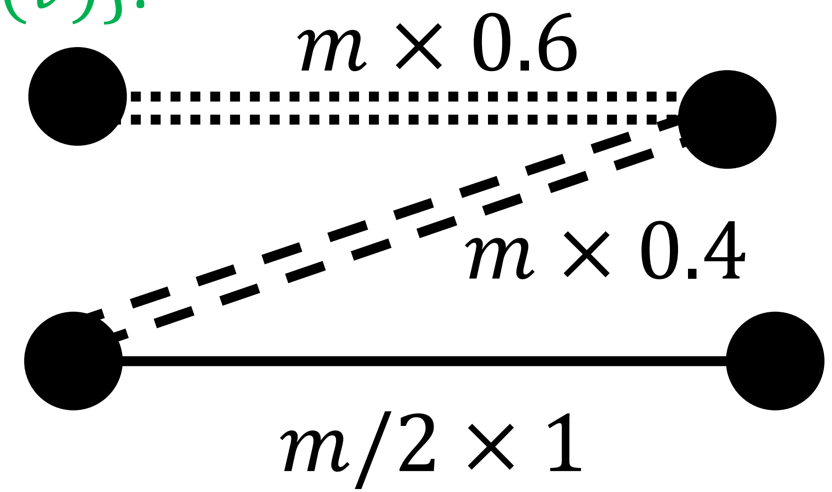
  (Edges = items, colors = bins).
- Many other generalizations: See my thesis!
  Geometric Bin Packing [Bansal-K. ,SODA'14],
  Vector Packing [Bansal-Elias-K. ,SODA'16] etc.





Weighted Bipartite Edge Coloring: Previous Works

- Conjecture 1. [Chung & Ross1991]
  There is a proper weighted coloring with $2m$ -1 colors where
  $m = \max_{\{v \in V\}}\{\min \# \; bins \; to \; pack \; w_e's \mid e \in \delta(v)\}.$



Lower bound:

- Ngo -Vu SODA'03 : $1.25\, m$

Upper bound:

- Du et al *SIAM J. Comp. '98*: $41m/16 = 2.562\, m$
- Correa-Goemans *STOC '04*: $2.548\, m$
- Feige-Singh *ESA '08*: $9m/4 = 2.25\, m$

This talk:

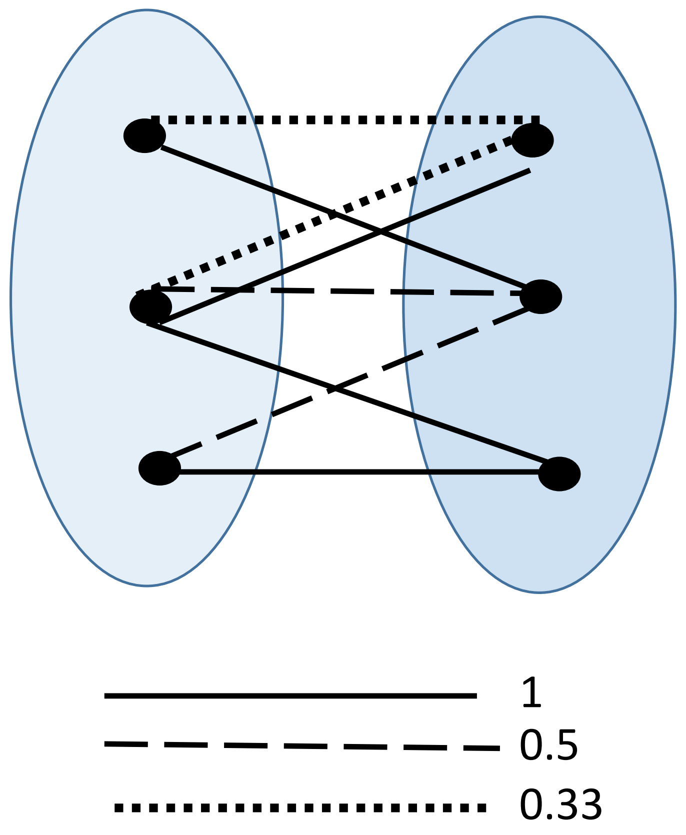
- $m = \max_{\{v \in V\}} \{\min \#\ bins\ to\ pack\ w_e's \mid e \in \delta(v)\}$

- Theorem 1: Polynomial time algorithm for proper edge coloring with $\frac{20}{9}m$ colors.
  - Purely combinatorial algorithm. (Coloring – Konig's theorem)
  - Intricate analysis using configuration linear program. (Bin Packing)
- Theorem 2: Polynomial time algorithm for proper edge coloring with $\frac{11}{5}m$ colors when all edge weights are $> \frac{1}{4}$.

Algorithm:

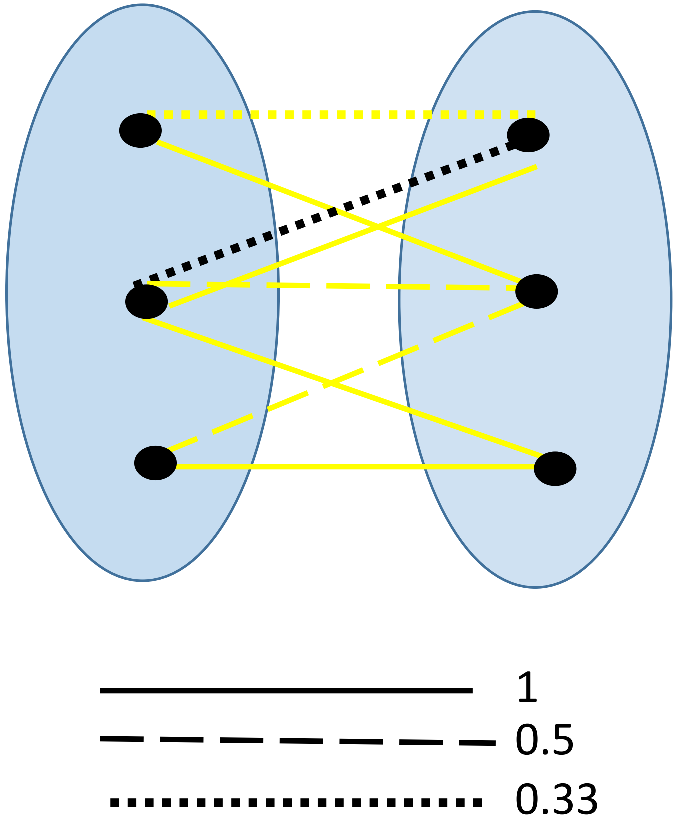
- 1. Start with an empty set.

$$F \leftarrow \emptyset.$$



Algorithm:

- 1. Include edges with weight $> \frac{1}{10}$ in F in non-increasing order of weight s.t. $\deg_F v \leq \lceil tm \rceil \forall\, v \in V$.



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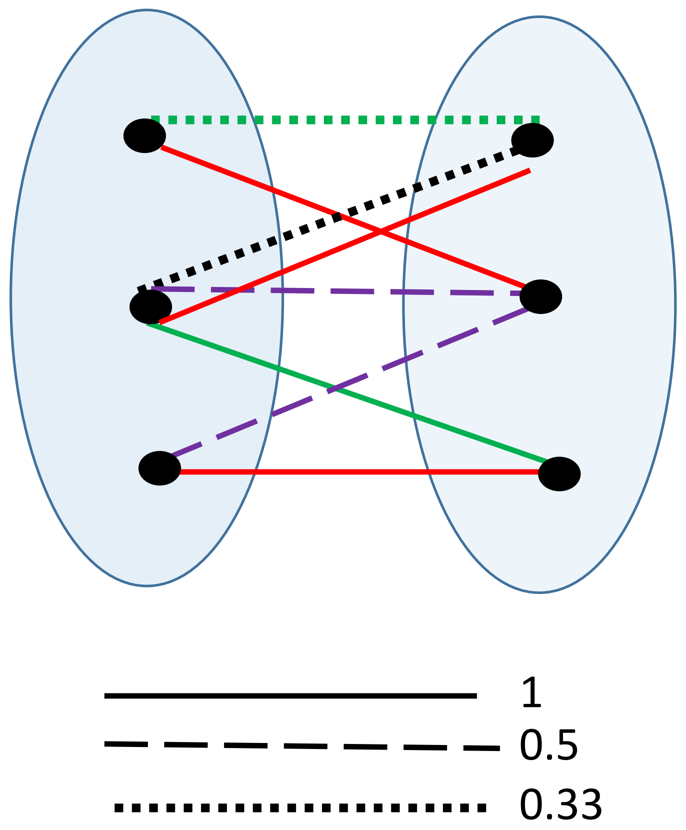
- 1. Include edges with weight $> \frac{1}{10}$ in F in non-increasing order of weight s.t. $\deg_F v \leq \lceil tm \rceil \forall\, v \in V$.
- 2. Decompose F into $r = \lceil\, tm \,\rceil$ matchings and color them using r colors [Konig's Thorem].



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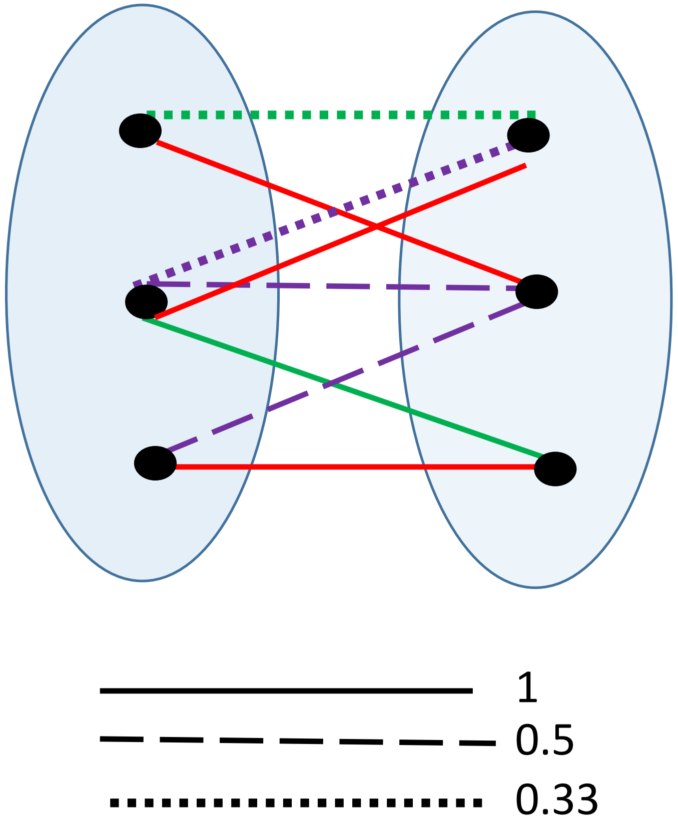
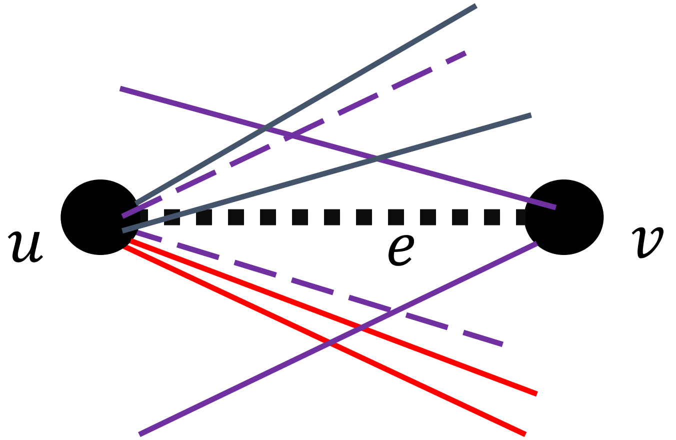
- 3. Greedily add remaining edges in non-increasing order of weight maintaining that the weighted degree of each color at each vertex is at most one [First Fit].



Proof of correctness: $t = 20/9$ is enough!

- Assume there is an edge $e := (u, v)$ with weight $w_e = \alpha$ that can not be added.
- Either u or v has degree $\geq tm$.
- Tight bin at v: weight $> 1 - \alpha$.
- Assume $deg(u) \geq tm$ and βm bins are tight on v.



Analysis: t> 20/9

- Edges incident at u or v can not be packed into m bins.
- More involved analysis using two Bin Packing Configuration LP and Dual LP.
- Number of bins $\geq$ Opt soln of Configuration LP Relaxation $\geq$ Dual Optimum $\geq$ Dual Feasible Solution $>$ m

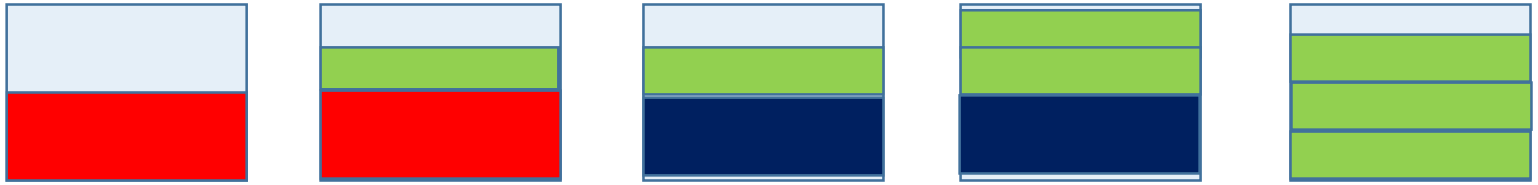
Analysis: t> 20/9 at vertex u

- There can be many item sizes. We discretize!
- Classify edges incident at u into three classes:
- **L**ARGE: (1/2,1], **M**EDIUM: (1/3,1/2], **S**MALL: (1/4, 1/3].
- Tight Bins: Bins with weight $> 1 - \alpha$.
- Lemma: All tight bins in our algorithm will have at most one item from $L \cup \mathrm{M}$.

Analysis: t> 20/9, at vertex u

- Possible Configurations of Tight Bins in **ALGO**:

(L) (L,S) (M,S) (M,S,S) (S,S,S)



- (L,M),(M,M),(M,M,S) does not appear in ALGO.
- Configurations in **OPT** packing are the following (or subsets of the following) :
- (L,M), (L, S), (M,M,S), (M, S, S), (S, S, S).
- Using valid configurations in OPT we need to cover all items in L, M, S.

Configuration LP

- Possible Configurations of Tight Bins in ALGO:
  x_1 bins: (L),
  x_2 bins: (L,S),
  x_3 bins: (M,S),
  x_4 bins: (M,S,S),
  x_5 bins: (S,S,S).
- Say, in OPT solution, there are
  y_1 bins: (L,M),
  y_2 bins: (L,S),
  y_3 bins: (M,M,S),
  y_4 bins: (M,S,S),
  y_5 bins: (S,S,S).

$$\min \sum_{i=1}^{5} y_i$$

$$\begin{aligned}
x_1 + x_2 + x_3 + x_4 + x_5 &\geq \tau \\
y_1 + y_2 &\geq x_1 + x_2 + z_1 \\
y_1 + 2y_3 + y_4 &\geq x_3 + x_4 + z_2 \\
y_2 + y_3 + 2y_4 + 3y_5 &\geq x_2 + x_3 + 2x_4 + 3x_5 + z_3 \\
z_1 + z_2 + z_3 &\geq \theta
\end{aligned}$$

Analysis:

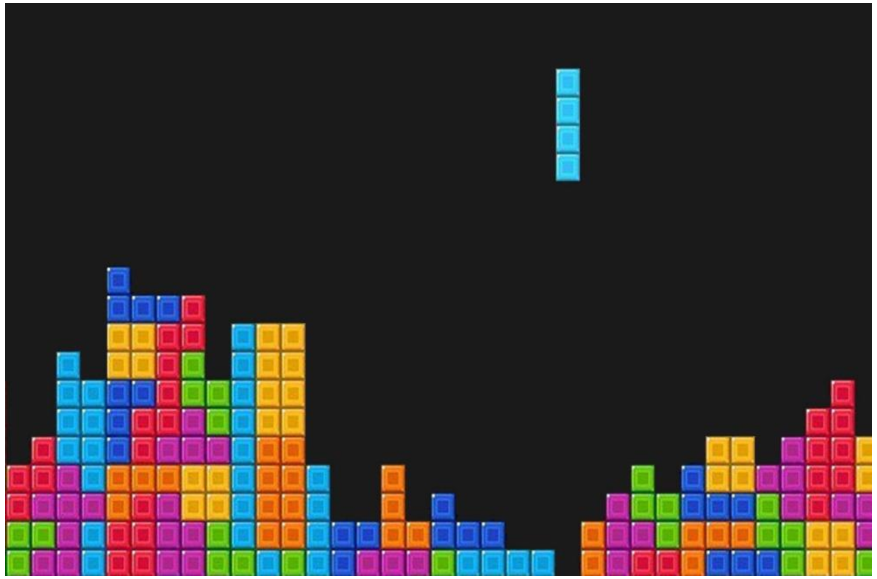
- For side v, more intricate analysis as there can be edges $< \alpha$!

- Dual optima for u : $\mathrm{D_u} > \frac{2tm}{3} - \frac{\beta m}{3}$.
- Dual optima for v: $\mathrm{D_v} > \frac{9\beta m}{13}$.
- If $t > 20/9$ then either $\mathrm{D_u}$ or $\mathrm{D_v}$ $is > m$.
- Giving us the desired contradiction.

Open Questions!

1. Resolving Chung-Ross conjecture.
   - Improve existential upper bound (20/9) or lower bound (5/4).
2. Better Approximation.
3. Online setting:
   - Present upper bound $5n$ (Correa-Goemans),
   - Lower bound $3n - 2$ (Tsai, Wang, Hwang).

Questions!



Extra Slides

- Online Setting, General Graphs, Knapsack version.
- Profit is arbitrary and total weight/bins is n and we aim to get 1/2n of the total profit.

Configuration LP

- $\mathbb{C}$*: set of configurations(possible way of feasibly packing a bin).*

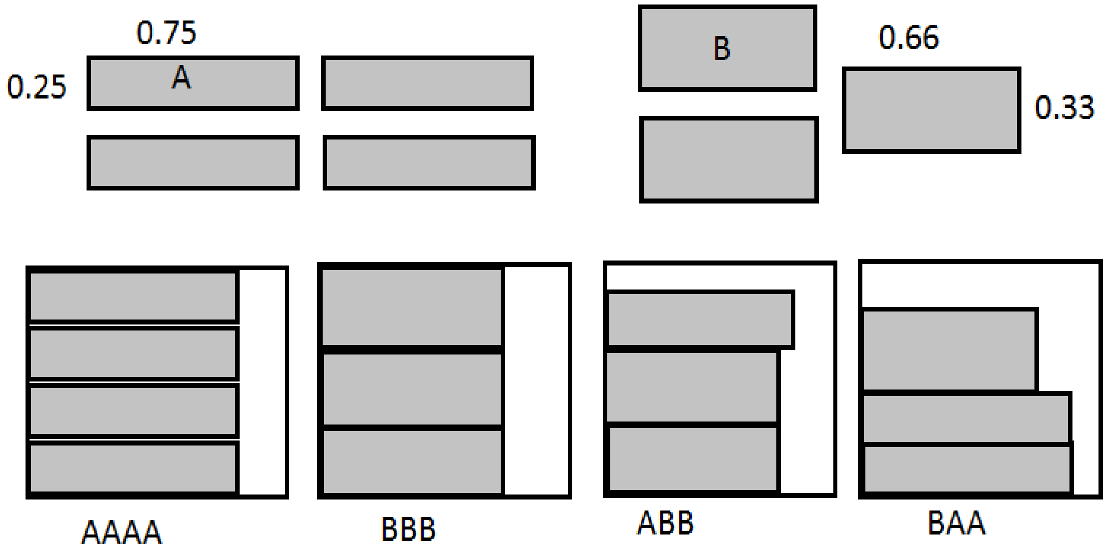
Primal:

$$\min \left\{ \sum_{C} x_C : \sum_{C \ni i} x_C \geq 1 \; (i \in I), x_C \geq 0 \; (C \in \mathbb{C}) \right\}$$

Objective: min # configurations(bins)

Constraint:

For each item, at least one configuration containing the item should be selected.



Configuration LP

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Primal:

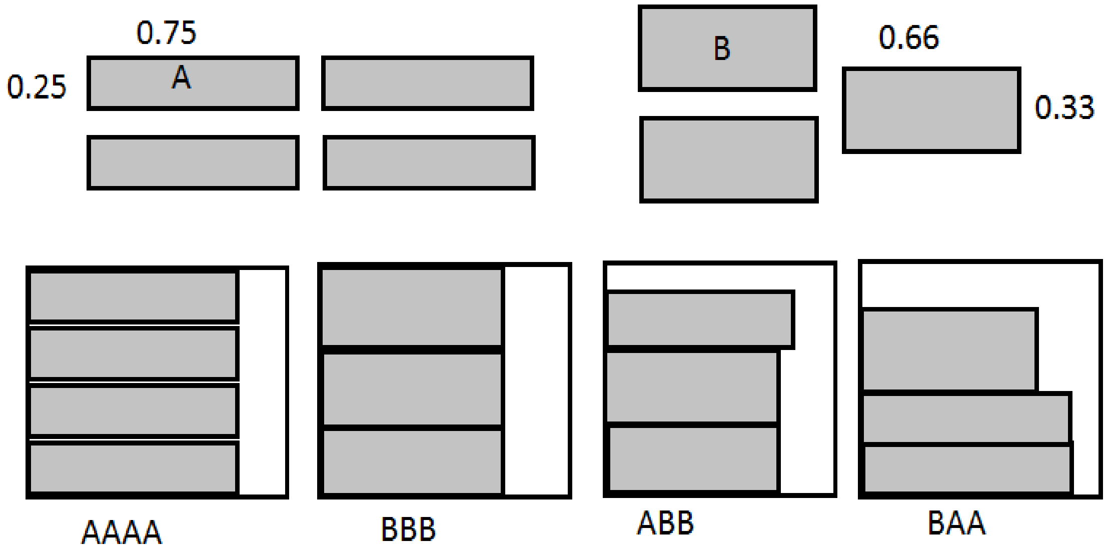
$$\min \{ \sum_{C} x_C : \sum_{C \ni i} x_C \geq 1 \; (i \in I), x_C \geq 0 \; (C \in \mathbb{C}) \}$$

Gilmore Gomory LP for multiple identical items:

$Min \{1^T x: \; Ax \geq b, \, x_C \geq 0 (C \in \mathbb{C})\}$

Columns: Feasible configurations

Rows: Items (or types of items)



Configuration LP

Gilmore Gomory LP:

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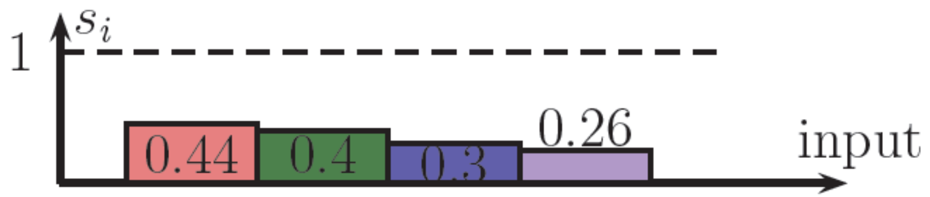

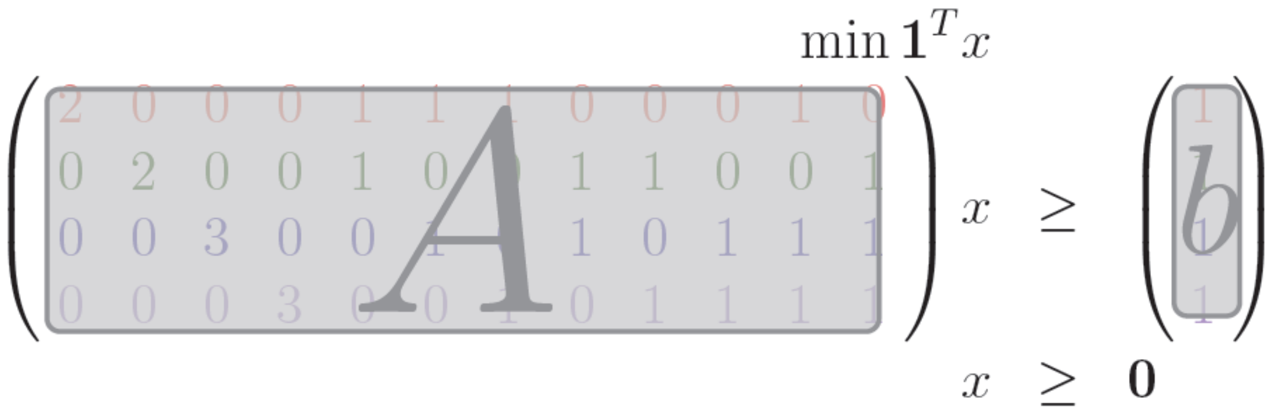

Rothvoss '13

Configuration LP

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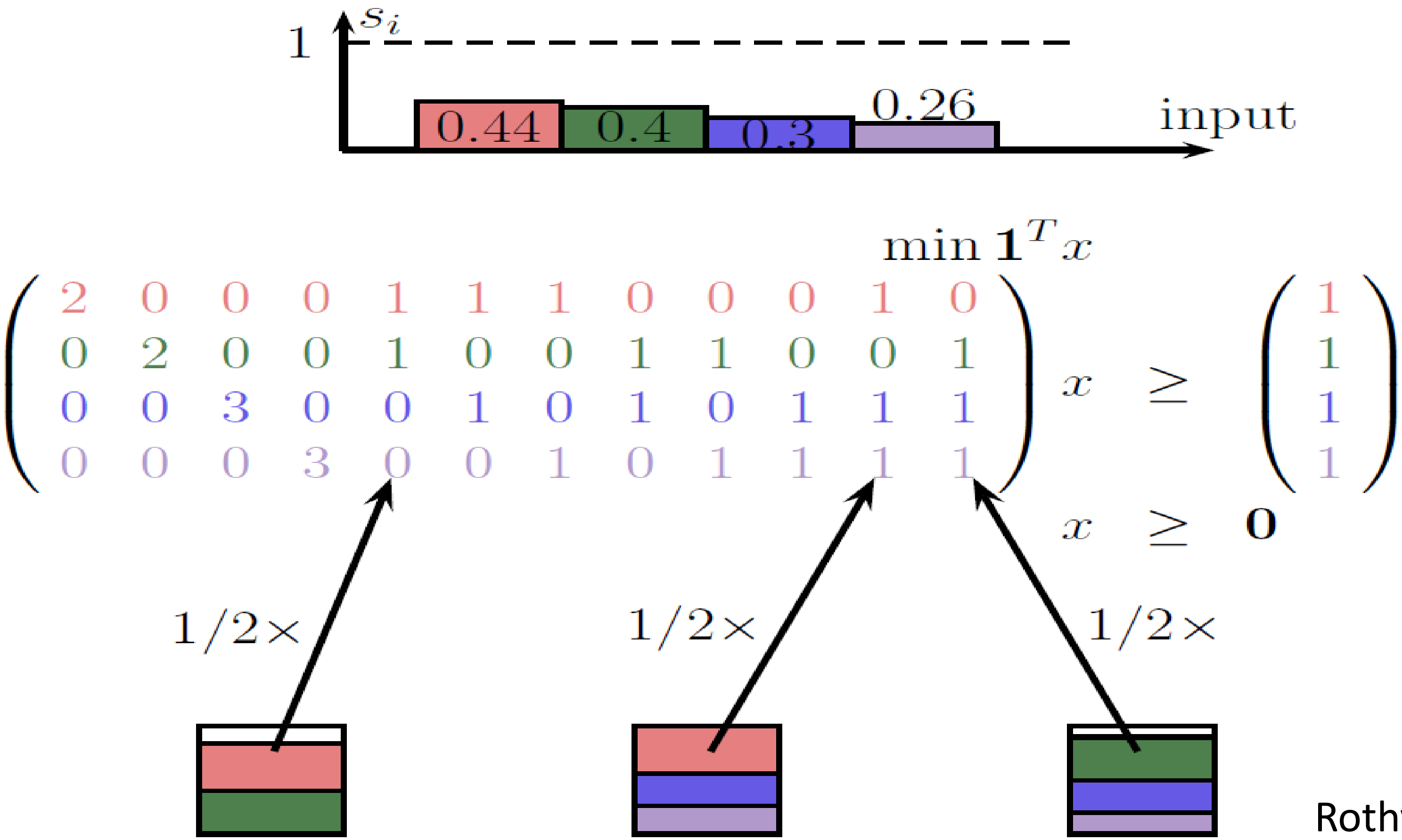

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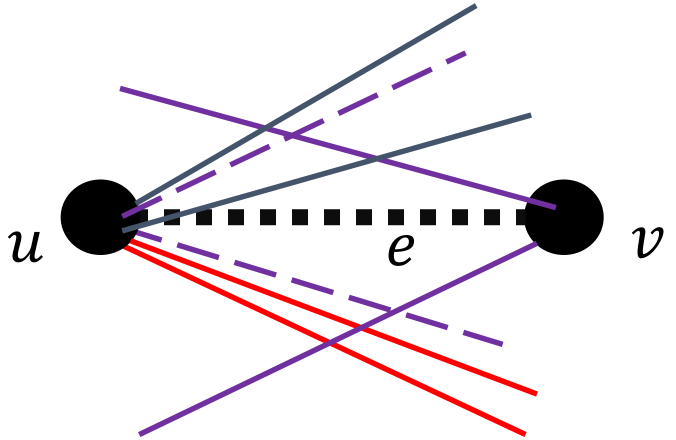
Proof of correctness: $t = 20/9$ is enough!

- Assume there is an edge $e := (u, v)$ with weight $w_e = \alpha$ that can not be added.
- Step 1: Include edges in F in non-dec order of weight s.t. $\deg_F v \leq \lceil tm \rceil \forall\, v \in V$.
- As e was not added in step 1, one of its endpoints have degree $\lceil tm \rceil$.
- Step 3: Greedily add remaining edges into maintaining that the weighted degree of each color at each vertex is at most one.
- *As* e was not added in step 4, $\forall$ color class C either $weight_C(u) > 1 - \alpha$ or $weight_C(v) > 1 - \alpha$.



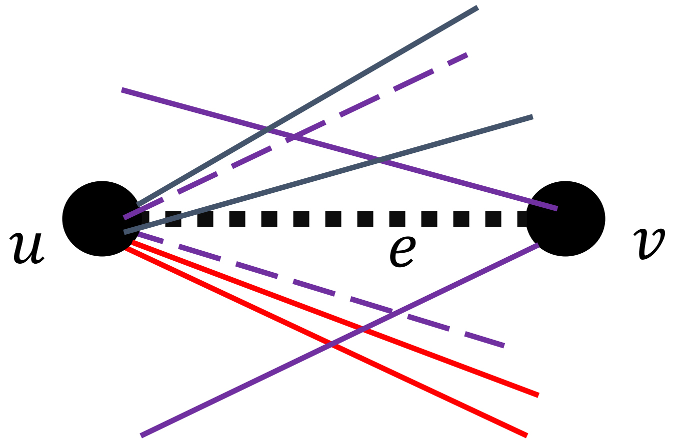
Tight bin: weight $> 1 - \alpha$.
Assume $deg(u) \geq tm$ and βm bins are tight on v.

Proof of correctness: $t = 20/9$ *is enough*!

- $\alpha < 1/3$
  - Each bin can contain at most two edges with weight >1/3.
  - As all edges incident to a vertex can be packed into m bins, there can be at most $2m$ edges incident to a vertex with weight > 1/3.
  - As we chose $t > 2$, $\alpha < 1/3$.
- $m > \beta m(1 - \alpha)$ [From v]

  $\Rightarrow 1 > \beta(1 - \alpha)$
- $m > (tm - \beta m)(1 - \alpha) + \beta m\alpha$ [From u]

  $\Rightarrow 1 > t(1 - \alpha) + \beta(2\alpha - 1)$



Tight bin: weight $> 1 - \alpha$.
Assume $deg(u) \geq tm$ and βm bins are tight on v.
Each bin at u has weight $> \alpha$

Analysis: t> 20/9

- Case A: $\alpha \leq \frac{1}{4}$.
- $t(1 - \alpha) + \beta(2\alpha - 1) \geq 1 \rightarrow$ Contradiction!
- Case B: $\frac{1}{4} < \alpha \leq \frac{1}{3}$.
- Edges incident at u or v can not be packed into m bins.
- More involved analysis using Bin Packing Configuration LP and Dual LP.
- Number of bins $\geq$ Opt soln of Configuration LP Relaxation $\geq$ Dual Optimum $\geq$ Dual Feasible Solution $>$ m

Analysis: t> 20/9 and $\frac{1}{4} < \mu \leq \frac{1}{3}$, at vertex u

- Classify edges incident at u into three classes:
- **L**ARGE: (1/2,1], **M**EDIUM: (1/3,1/2], **S**MALL: (1/4, 1/3].
- Observation: Each configuration *(feasible way of packing a bin)* will have ≤ 1 items from L, ≤ 2 items from $L \cup \mathrm{M}$ and ≤ 3 items from $L \cup \mathrm{M} \cup S$.
- Tight Bins: Bins with weight $> 1 - \alpha$.
- Open Bins: Bins with weight $\in (0, 1 - \alpha]$
- Lemma: All tight bins in our algorithm will have at most one item from $L \cup \mathrm{M}$.

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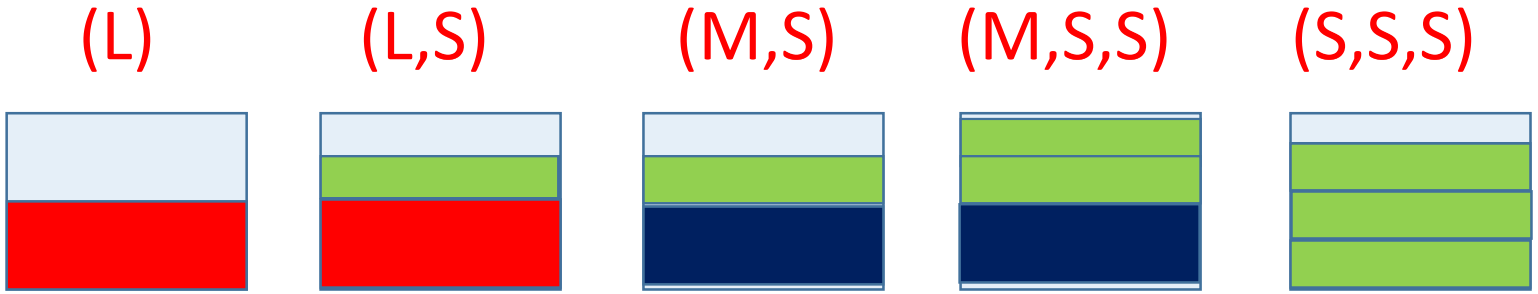
- Possible Configurations of Tight Bins in ALGO:



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- Possible Configurations of Tight Bins in ALGO:
  x_1 bins: (L), x_2 bins: (L,S), x_3 bins: (M,S), x_4 bins: (M,S,S), x_5 bins: (S,S,S).
- Let z_1, z_2, z_3 be the number of items of type L, M, S in open bins.
- $x_1 + x_2 + x_3 + x_4 + x_5 = Number\ of\ tight\ bins = \tau \geq (tm - \beta m)$.
- Using valid configurations in OPT we need to cover all items in L, M, S.
- Say, in OPT solution, there are y_1 bins: (L,M), y_2 bins: (L,S), y_3 bins: (M,M,S), y_4 bins: (M,S,S), y_5 bins: (S,S,S).
- Number of items in L,M,S comes as a function of x_i ‘s.
- e.g., For L items: $y_1 + y_2 \geq x_1 + x_2 + z_1$
- This gives us the following LP.

Configuration LP

- Possible Configurations of Tight Bins in ALGO: x_1 bins: (L), x_2 bins: (L,S), x_3 bins: (M,S), x_4 bins: (M,S,S), x_5 bins: (S,S,S).
- Let z_1, z_2, z_3 be the number of items of type L, M, S in open colors.
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$$min \sum_{i=1}^{5} y_i$$

$$\begin{aligned}
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z_1 + z_2 + z_3 &\geq \theta
\end{aligned}$$

Configuration LP

- $\mathbb{C}$*: set of configurations in OPT,*
- *T: types of items* (L, M, S).

Primal:

$$\min \{ \sum_{C} x_C : \sum_{C \ni i} x_C \geq n_i \ (i \in T), x_C \geq 0 \ (C \in \mathbb{C}) \}$$

Dual:

$$\max \{ \sum_{i \in I} n_i v_i : \sum_{i \in C} v_i \leq 1 \ (C \in \mathbb{C}), v_i \geq 0 (i \in T) \}$$

Analysis:

- For side v, more intricate analysis as there can be edges $< \alpha$!

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