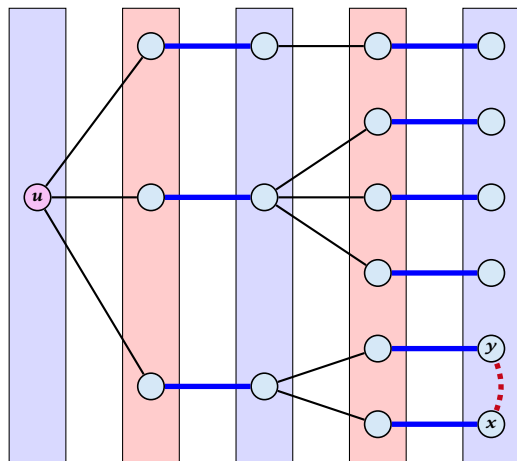


# How to find an augmenting path?

Construct an alternating tree.



even nodes

odd nodes

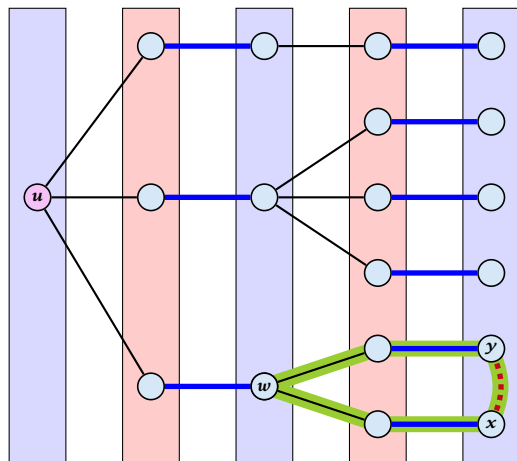
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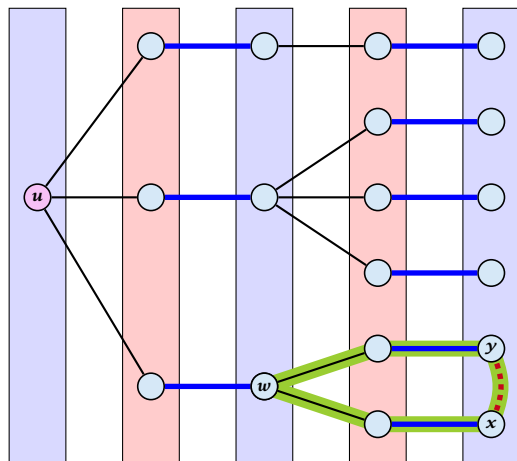
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The cycle  $w \leftrightarrow y - x \leftrightarrow w$   
is called a **blossom**.

$w$  is called the **base** of the  
blossom (even node!!!).

The path  $u-w$  is called the  
**stem** of the blossom.

# Flowers and Blossoms

## Definition 1

A **flower** in a graph  $G = (V, E)$  w.r.t. a matching  $M$  and a (free) root node  $r$ , is a subgraph with two components:

- ▶ A **stem** is an even length alternating path that starts at the root node  $r$  and terminates at some node  $w$ . We permit the possibility that  $r = w$  (empty stem).
- ▶ A **blossom** is an odd length alternating cycle that starts and terminates at the terminal node  $w$  of a stem and has no other node in common with the stem.  $w$  is called the **base** of the blossom.

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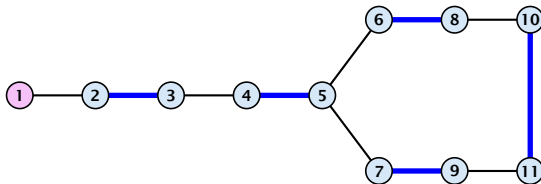
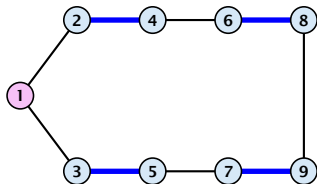
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# Flowers and Blossoms



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## Properties:

1. A stem spans  $2\ell + 1$  nodes and contains  $\ell$  matched edges for some integer  $\ell \geq 0$ .
2. A blossom spans  $2k + 1$  nodes and contains  $k$  matched edges for some integer  $k \geq 1$ . The matched edges match all nodes of the blossom except the base.
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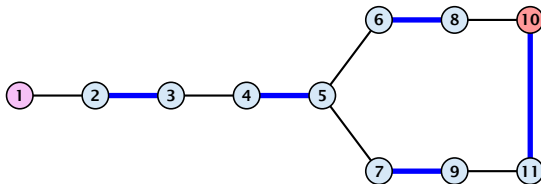
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# Shrinking Blossoms

When during the alternating tree construction we discover a blossom  $B$  we replace the graph  $G$  by  $G' = G/B$ , which is obtained from  $G$  by contracting the blossom  $B$ .

- ▶ Delete all vertices in  $B$  (and its incident edges) from  $G$ .
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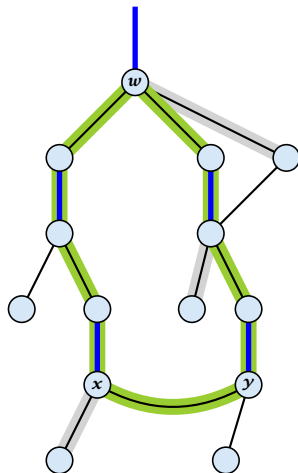
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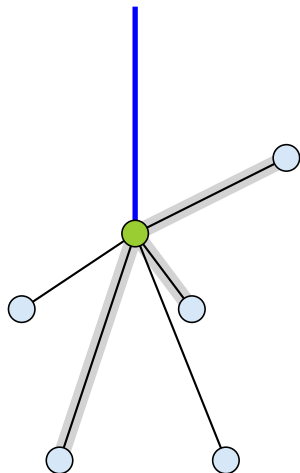
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- ▶ Edges of  $T$  that connect a node  $u$  not in  $B$  to a node in  $B$  become tree edges in  $T'$  connecting  $u$  to  $b$ .
- ▶ Matching edges (there is at most one) that connect a node  $u$  not in  $B$  to a node in  $B$  become matching edges in  $M'$ .
- ▶ Nodes that are connected in  $G$  to at least one node in  $B$  become connected to  $b$  in  $G'$ .

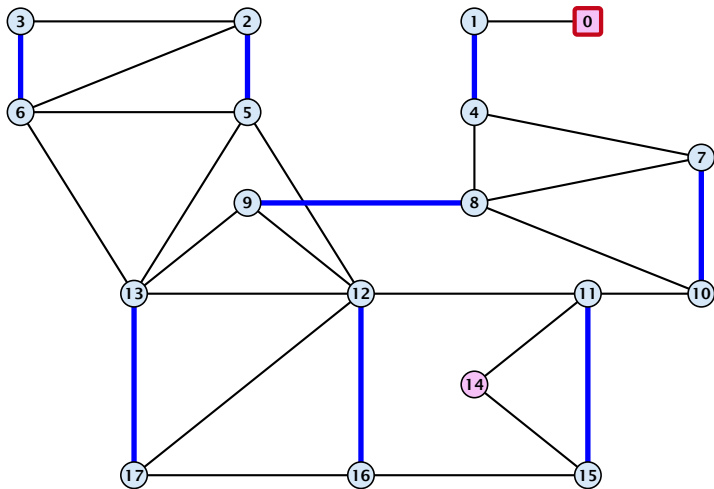


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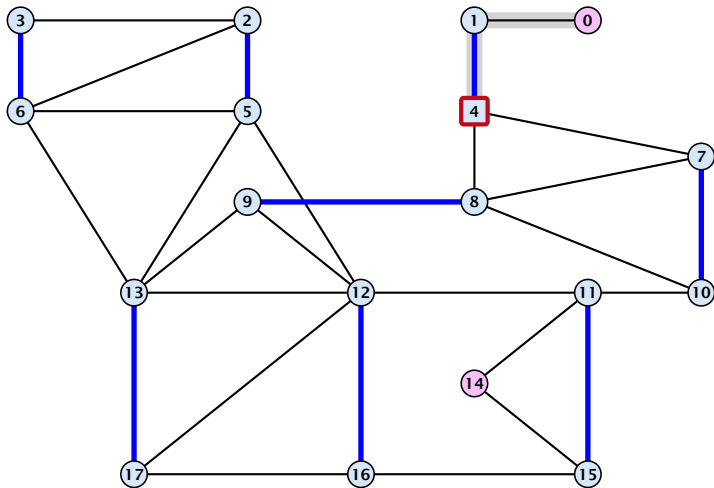
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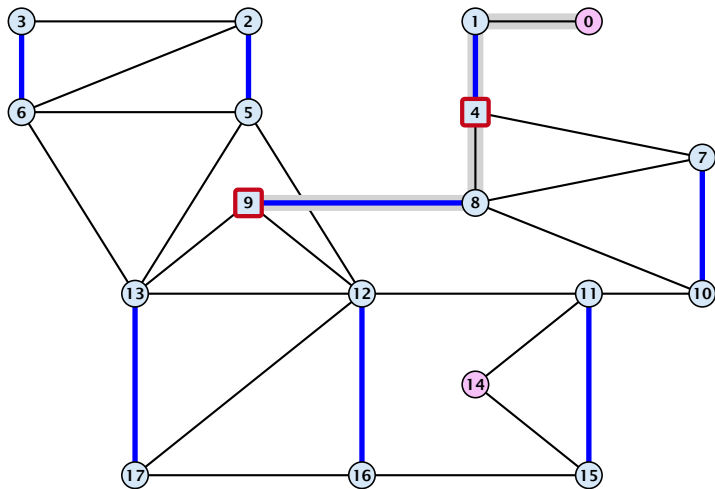
# Example: Blossom Algorithm



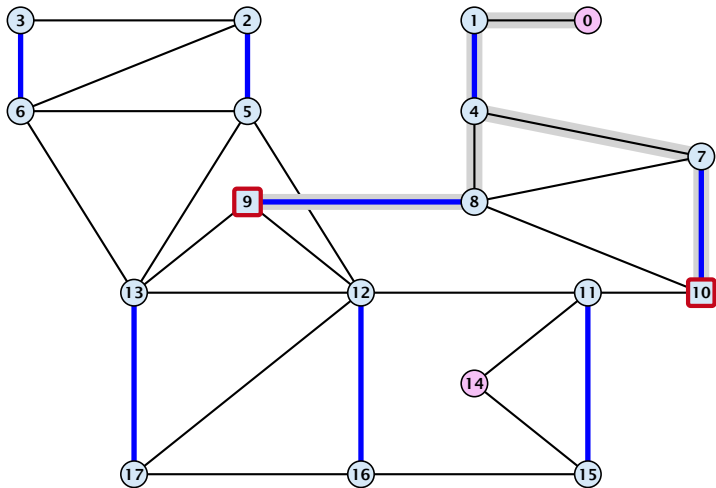
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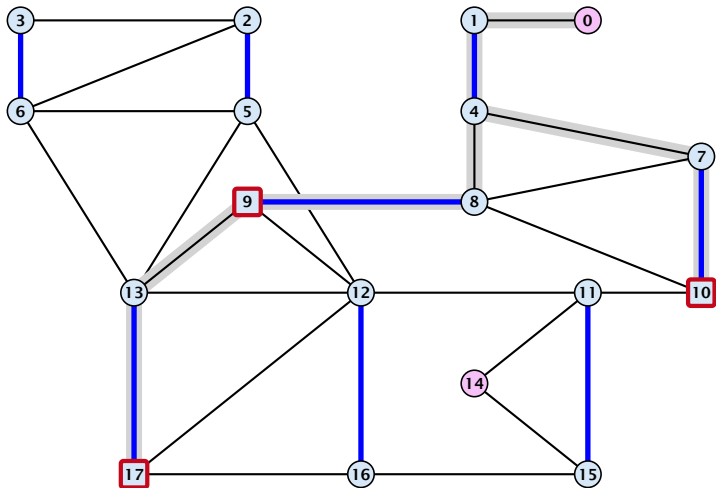
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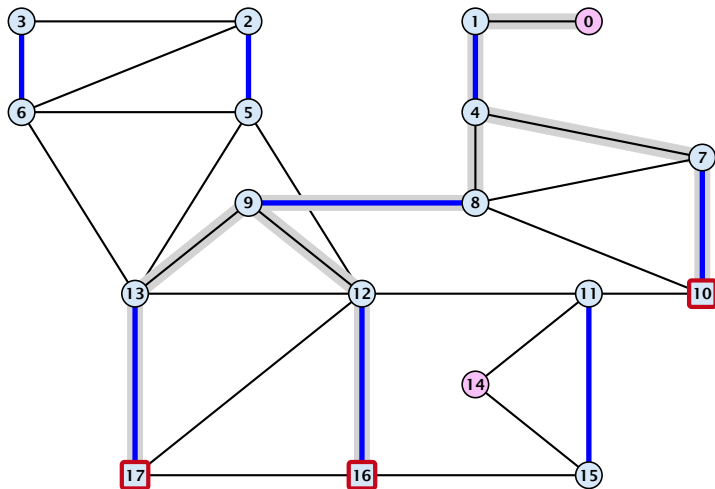
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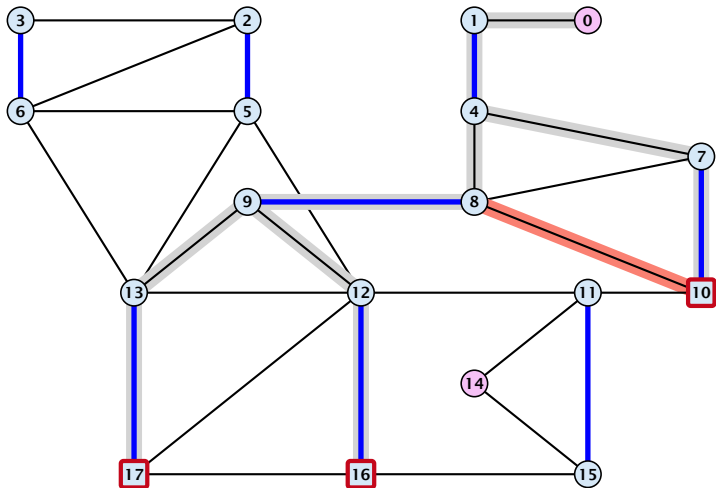


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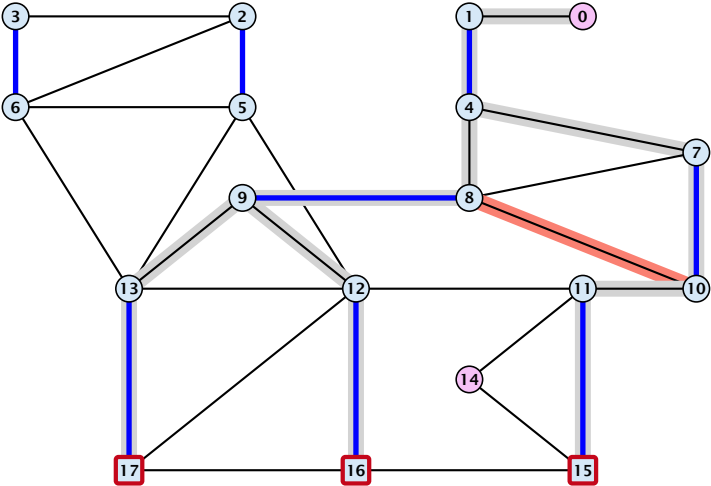




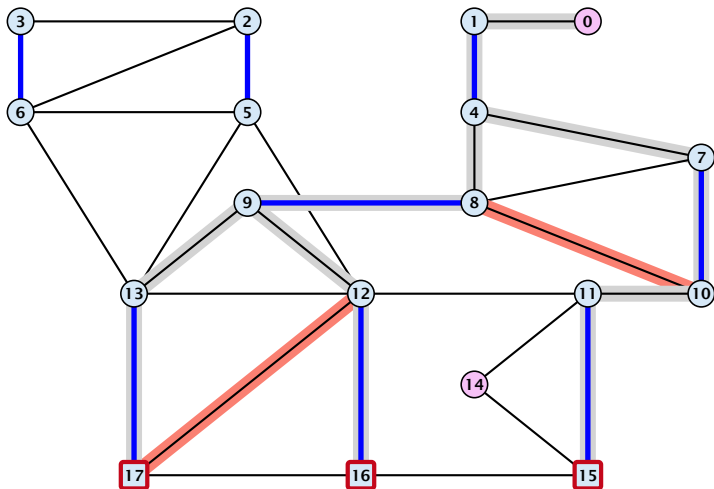
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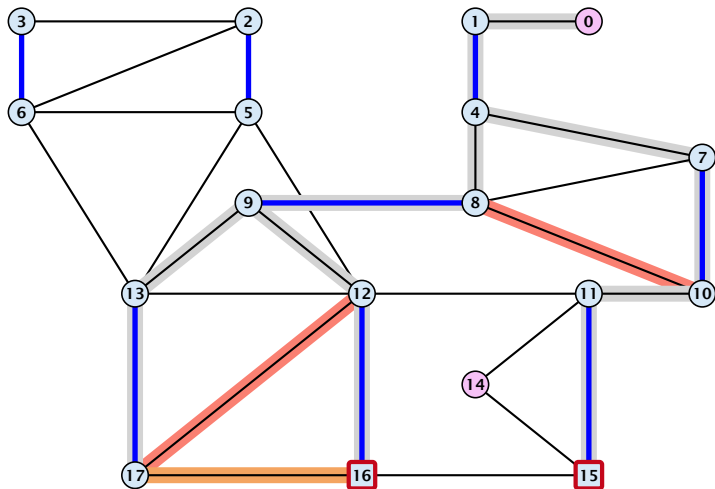
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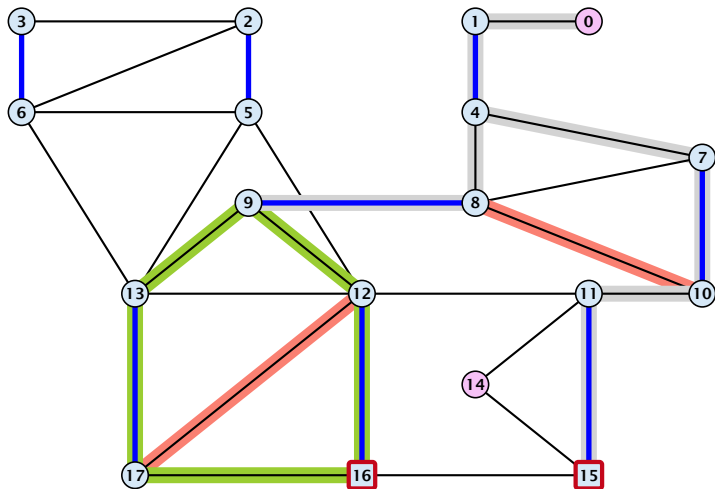
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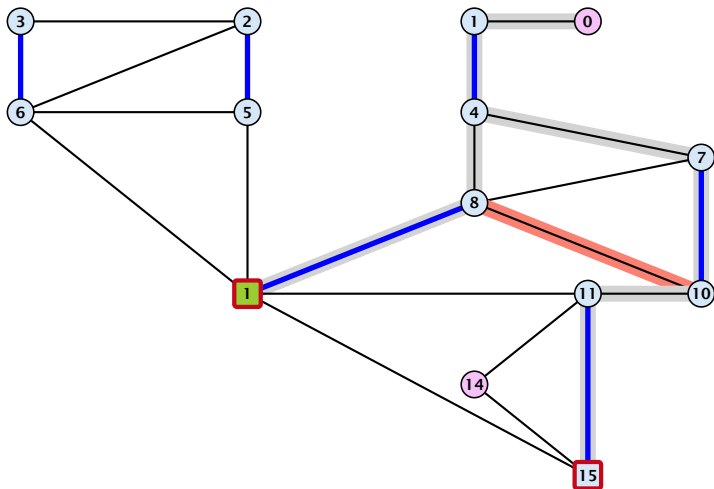
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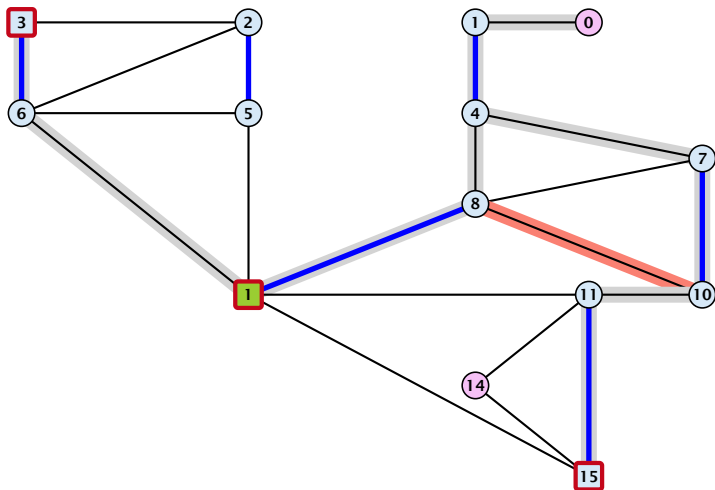
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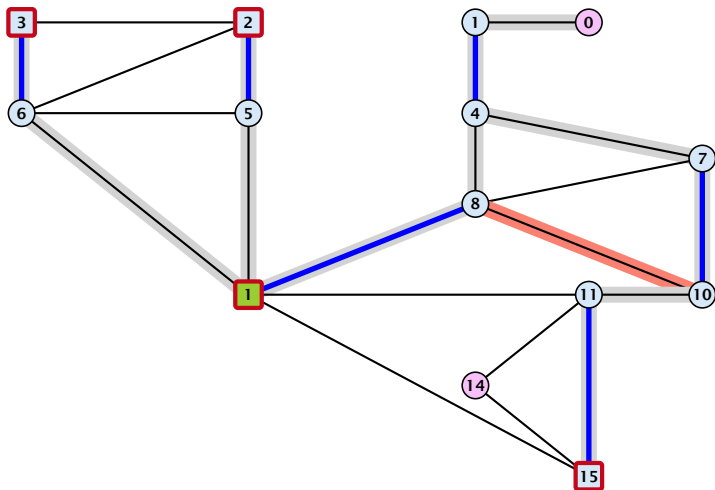
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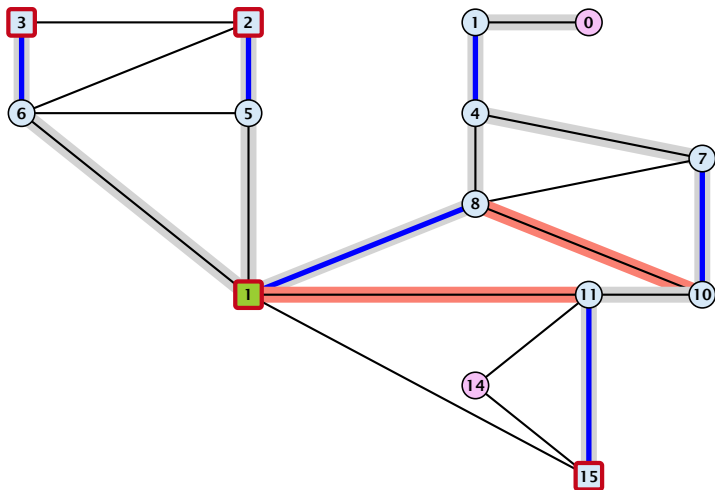


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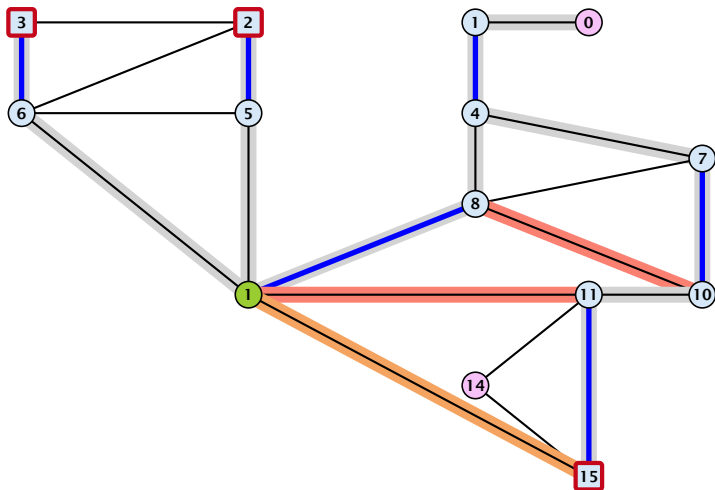




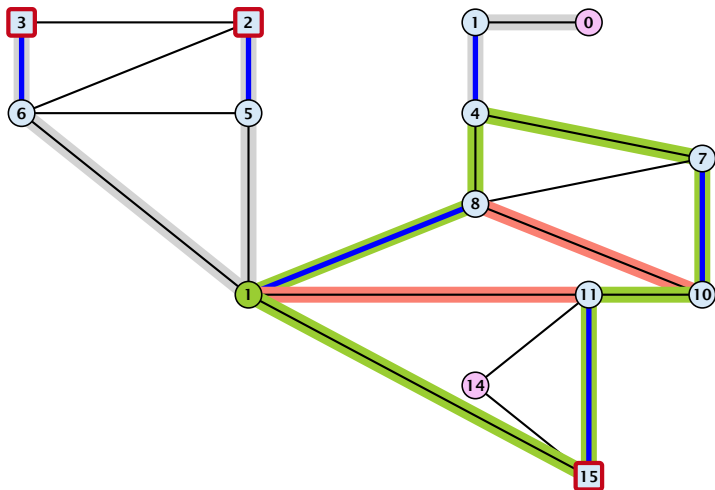
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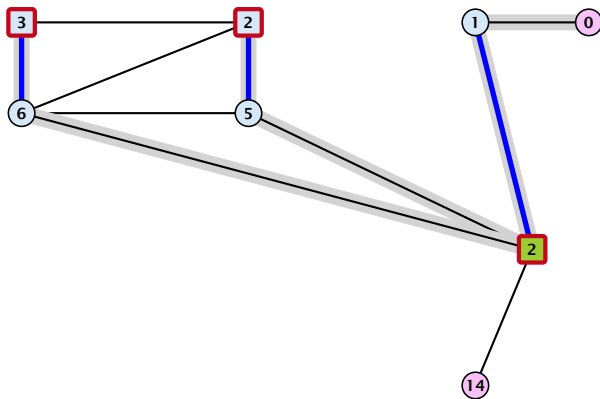
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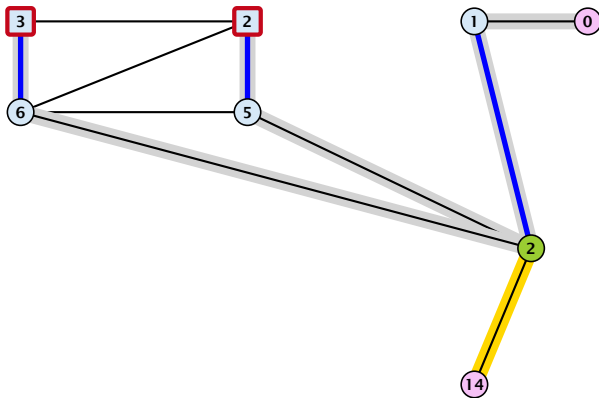
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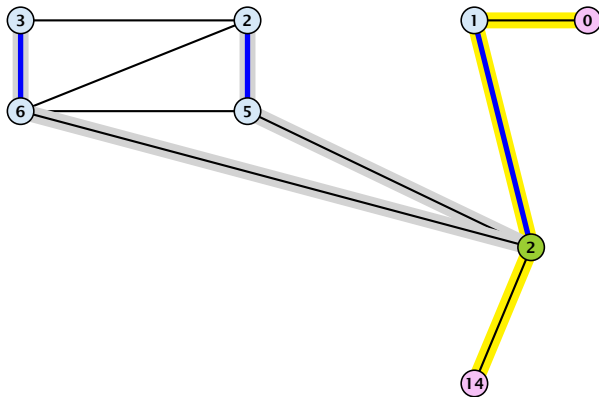
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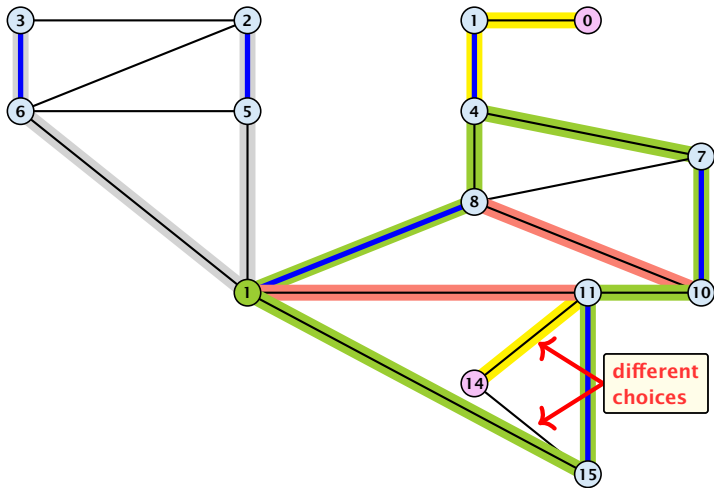
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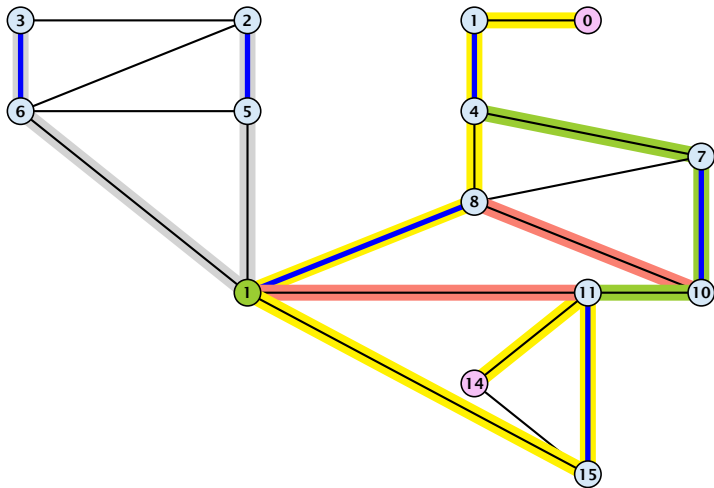
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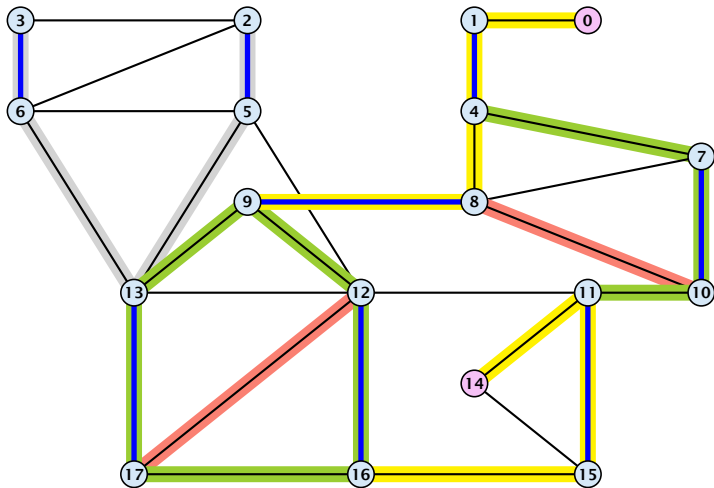


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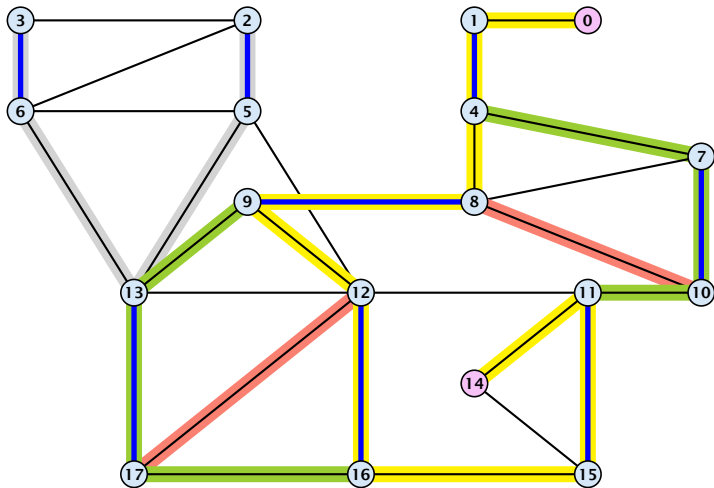




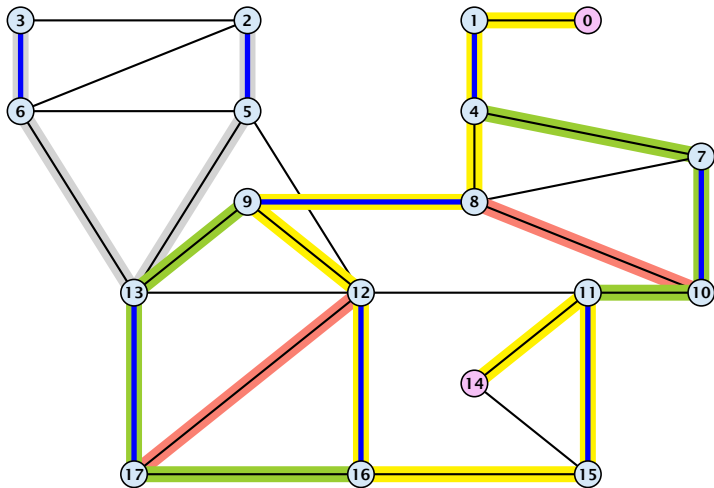
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# Correctness

Assume that in  $G$  we have a flower w.r.t. matching  $M$ . Let  $r$  be the root,  $B$  the blossom, and  $w$  the base. Let graph  $G' = G/B$  with pseudonode  $b$ . Let  $M'$  be the matching in the contracted graph.

## Lemma 2

*If  $G'$  contains an augmenting path  $P'$  starting at  $r$  (or the pseudo-node containing  $r$ ) w.r.t. the matching  $M'$  then  $G$  contains an augmenting path starting at  $r$  w.r.t. matching  $M$ .*

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- ▶ Next suppose that the stem is non-empty.

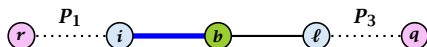
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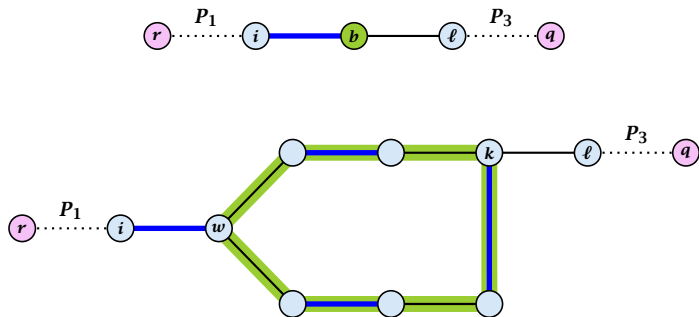
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# Correctness

- ▶ After the expansion  $\ell$  must be incident to some node in the blossom. Let this node be  $k$ .
- ▶ If  $k \neq w$  there is an alternating path  $P_2$  from  $w$  to  $k$  that ends in a matching edge.
- ▶  $P_1 \circ (i, w) \circ P_2 \circ (k, \ell) \circ P_3$  is an alternating path.
- ▶ If  $k = w$  then  $P_1 \circ (i, w) \circ (w, \ell) \circ P_3$  is an alternating path.

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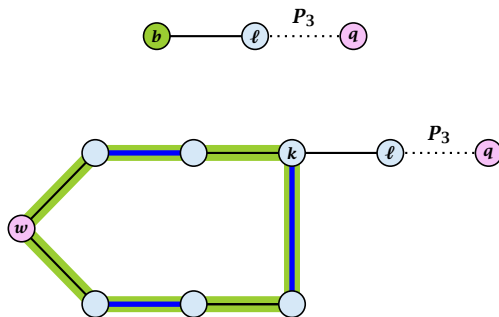


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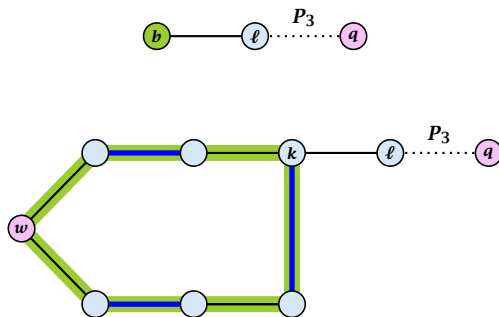


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- ▶ The path  $r \circ P_2 \circ (k, l) \circ P_3$  is an alternating path.

## Lemma 3

*If  $G$  contains an augmenting path  $P$  from  $r$  to  $q$  w.r.t. matching  $M$  then  $G'$  contains an augmenting path from  $r$  (or the pseudo-node containing  $r$ ) to  $q$  w.r.t.  $M'$ .*

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- ▶ If  $P$  does not contain a node from  $B$  there is nothing to prove.
- ▶ We can assume that  $r$  and  $q$  are the only free nodes in  $G$ .

## Case 1: empty stem

Let  $i$  be the last node on the path  $P$  that is part of the blossom.

$P$  is of the form  $P_1 \circ (i, j) \circ P_2$ , for some node  $j$  and  $(i, j)$  is unmatched.

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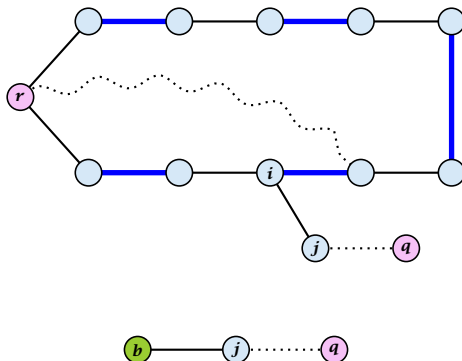
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# Correctness

## Illustration for Case 1:



## Correctness

### Case 2: non-empty stem

Let  $P_3$  be alternating path from  $r$  to  $w$ ; this exists because  $r$  and  $w$  are root and base of a blossom. Define  $M_+ = M \oplus P_3$ .

In  $M_+$ ,  $r$  is matched and  $w$  is unmatched.

$G$  must contain an augmenting path w.r.t. matching  $M_+$ , since  $M$  and  $M_+$  have same cardinality.

This path must go between  $w$  and  $q$  as these are the only unmatched vertices w.r.t.  $M_+$ .

For  $M'_+$  the blossom has an empty stem. Case 1 applies.

$G'$  has an augmenting path w.r.t.  $M'_+$ . It must also have an augmenting path w.r.t.  $M'$ , as both matchings have the same cardinality.

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This path must go between  $w$  and  $q$  as these are the only unmatched vertices w.r.t.  $M_+$ .

For  $M'_+$  the blossom has an empty stem. Case 1 applies.

$G'$  has an augmenting path w.r.t.  $M'_+$ . It must also have an augmenting path w.r.t.  $M'$ , as both matchings have the same cardinality.

This path must go between  $r$  and  $q$ .



## Correctness

### Case 2: non-empty stem

Let  $P_3$  be alternating path from  $r$  to  $w$ ; this exists because  $r$  and  $w$  are root and base of a blossom. Define  $M_+ = M \oplus P_3$ .

In  $M_+$ ,  $r$  is matched and  $w$  is unmatched.

$G$  must contain an augmenting path w.r.t. matching  $M_+$ , since  $M$  and  $M_+$  have same cardinality.

This path must go between  $w$  and  $q$  as these are the only unmatched vertices w.r.t.  $M_+$ .

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This path must go between  $r$  and  $q$ .

**Algorithm 49**  $\text{search}(r, \text{found})$

- 1: set  $\bar{A}(i) \leftarrow A(i)$  for all nodes  $i$
- 2:  $\text{found} \leftarrow \text{false}$
- 3: unlabel all nodes;
- 4: give an even label to  $r$  and initialize  $\text{list} \leftarrow \{r\}$
- 5: **while**  $\text{list} \neq \emptyset$  **do**
- 6:     delete a node  $i$  from  $\text{list}$
- 7:      $\text{examine}(i, \text{found})$
- 8:     **if**  $\text{found} = \text{true}$  **then return**

Search for an augmenting path  
starting at  $r$ .

**Algorithm 49**  $\text{search}(r, \text{found})$

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- 8:     **if**  $\text{found} = \text{true}$  **then return**

$A(i)$  contains neighbours of node  $i$ .

We create a copy  $\bar{A}(i)$  so that we later  
can shrink blossoms.



**Algorithm 49**  $\text{search}(r, \text{found})$

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5: **while**  $\text{list} \neq \emptyset$  **do**

6:     delete a node  $i$  from  $\text{list}$

7:      $\text{examine}(i, \text{found})$

8:     **if**  $\text{found} = \text{true}$  **then return**

*found* is just a Boolean that allows  
to abort the search process...

**Algorithm 49**  $\text{search}(r, \text{found})$

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5: **while**  $\text{list} \neq \emptyset$  **do**

6:     delete a node  $i$  from  $\text{list}$

7:      $\text{examine}(i, \text{found})$

8:     **if**  $\text{found} = \text{true}$  **then return**

In the beginning no node is in the tree.

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3: unlabel all nodes;

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5: **while**  $\text{list} \neq \emptyset$  **do**

6:     delete a node  $i$  from  $\text{list}$

7:      $\text{examine}(i, \text{found})$

8:     **if**  $\text{found} = \text{true}$  **then return**

Put the root in the tree.

*list* could also be a set or a stack.

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2:  $\text{found} \leftarrow \text{false}$

3: unlabel all nodes;

4: give an even label to  $r$  and initialize  $\text{list} \leftarrow \{r\}$

5: **while**  $\text{list} \neq \emptyset$  **do**

6:     delete a node  $i$  from  $\text{list}$

7:      $\text{examine}(i, \text{found})$

8:     **if**  $\text{found} = \text{true}$  **then return**

As long as there are nodes with  
unexamined neighbours...

**Algorithm 49**  $\text{search}(r, \text{found})$

- 1: set  $\bar{A}(i) \leftarrow A(i)$  for all nodes  $i$
- 2:  $\text{found} \leftarrow \text{false}$
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- 4: give an even label to  $r$  and initialize  $\text{list} \leftarrow \{r\}$
- 5: **while**  $\text{list} \neq \emptyset$  **do**
- 6:     delete a node  $i$  from  $\text{list}$
- 7:     **examine** $(i, \text{found})$
- 8:     **if**  $\text{found} = \text{true}$  **then return**

...examine the next one

**Algorithm 49**  $\text{search}(r, \text{found})$

- 1: set  $\bar{A}(i) \leftarrow A(i)$  for all nodes  $i$
- 2:  $\text{found} \leftarrow \text{false}$
- 3: unlabel all nodes;
- 4: give an even label to  $r$  and initialize  $\text{list} \leftarrow \{r\}$
- 5: **while**  $\text{list} \neq \emptyset$  **do**
- 6:     delete a node  $i$  from  $\text{list}$
- 7:     examine( $i, \text{found}$ )
- 8:     **if**  $\text{found} = \text{true}$  **then return**

If you found augmenting path  
abort and start from next root.

**Algorithm 50** examine( $i$ ,  $found$ )

```
1: for all  $j \in \bar{A}(i)$  do  
2:   if  $j$  is even then contract( $i$ ,  $j$ ) and return  
3:   if  $j$  is unmatched then  
4:      $q \leftarrow j$ ;  
5:     pred( $q$ )  $\leftarrow i$ ;  
6:      $found \leftarrow \text{true}$ ;  
7:     return  
8:   if  $j$  is matched and unlabeled then  
9:     pred( $j$ )  $\leftarrow i$ ;  
10:    pred(mate( $j$ ))  $\leftarrow j$ ;  
11:    add mate( $j$ ) to  $list$ 
```

Examine the neighbours of a node  $i$

**Algorithm 50** examine( $i$ ,  $found$ )

```
1: for all  $j \in \bar{A}(i)$  do  
2:   if  $j$  is even then contract( $i$ ,  $j$ ) and return  
3:   if  $j$  is unmatched then  
4:      $q \leftarrow j$ ;  
5:     pred( $q$ )  $\leftarrow i$ ;  
6:      $found \leftarrow \text{true}$ ;  
7:     return  
8:   if  $j$  is matched and unlabeled then  
9:     pred( $j$ )  $\leftarrow i$ ;  
10:    pred(mate( $j$ ))  $\leftarrow j$ ;  
11:    add mate( $j$ ) to  $list$ 
```

For all neighbours  $j$  do...



**Algorithm 50** examine( $i, found$ )

```
1: for all  $j \in \bar{A}(i)$  do
2:   if  $j$  is even then contract( $i, j$ ) and return
3:   if  $j$  is unmatched then
4:      $q \leftarrow j$ ;
5:     pred( $q$ )  $\leftarrow i$ ;
6:      $found \leftarrow \text{true}$ ;
7:     return
8:   if  $j$  is matched and unlabeled then
9:     pred( $j$ )  $\leftarrow i$ ;
10:    pred(mate( $j$ ))  $\leftarrow j$ ;
11:    add mate( $j$ ) to  $list$ 
```

You have found a blossom...

**Algorithm 50** examine( $i, found$ )

```
1: for all  $j \in \bar{A}(i)$  do  
2:   if  $j$  is even then contract( $i, j$ ) and return  
3:   if  $j$  is unmatched then  
4:      $q \leftarrow j$ ;  
5:     pred( $q$ )  $\leftarrow i$ ;  
6:      $found \leftarrow \text{true}$ ;  
7:     return  
8:   if  $j$  is matched and unlabeled then  
9:     pred( $j$ )  $\leftarrow i$ ;  
10:    pred(mate( $j$ ))  $\leftarrow j$ ;  
11:    add mate( $j$ ) to list
```

You have found a free node which gives you an augmenting path.

**Algorithm 50** examine( $i, found$ )

```
1: for all  $j \in \bar{A}(i)$  do  
2:   if  $j$  is even then contract( $i, j$ ) and return  
3:   if  $j$  is unmatched then  
4:      $q \leftarrow j$ ;  
5:     pred( $q$ )  $\leftarrow i$ ;  
6:      $found \leftarrow \text{true}$ ;  
7:     return  
8:   if  $j$  is matched and unlabeled then  
9:     pred( $j$ )  $\leftarrow i$ ;  
10:    pred(mate( $j$ ))  $\leftarrow j$ ;  
11:    add mate( $j$ ) to list
```

If you find a matched node that is not  
in the tree you grow...

### Algorithm 50 examine( $i, found$ )

```
1: for all  $j \in \bar{A}(i)$  do
2:   if  $j$  is even then contract( $i, j$ ) and return
3:   if  $j$  is unmatched then
4:      $q \leftarrow j$ ;
5:     pred( $q$ )  $\leftarrow i$ ;
6:      $found \leftarrow \text{true}$ ;
7:     return
8:   if  $j$  is matched and unlabeled then
9:     pred( $j$ )  $\leftarrow i$ ;
10:    pred(mate( $j$ ))  $\leftarrow j$ ;
11:    add mate( $j$ ) to  $list$ 
```

$mate(j)$  is a new node from  
which you can grow further.

### Algorithm 51 $\text{contract}(i, j)$

- 1: trace pred-indices of  $i$  and  $j$  to identify a blossom  $B$
- 2: create new node  $b$  and set  $\bar{A}(b) \leftarrow \cup_{x \in B} \bar{A}(x)$
- 3: label  $b$  even and add to *list*
- 4: update  $\bar{A}(j) \leftarrow \bar{A}(j) \cup \{b\}$  for each  $j \in \bar{A}(b)$
- 5: form a circular double linked list of nodes in  $B$
- 6: delete nodes in  $B$  from the graph

Contract blossom identified by  
nodes  $i$  and  $j$

### Algorithm 51 $\text{contract}(i, j)$

- 1: trace pred-indices of  $i$  and  $j$  to identify a blossom  $B$
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- 5: form a circular double linked list of nodes in  $B$
- 6: delete nodes in  $B$  from the graph

Get all nodes of the blossom.

Time:  $\mathcal{O}(m)$

### Algorithm 51 $\text{contract}(i, j)$

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Identify all neighbours of  $b$ .

Time:  $\mathcal{O}(m)$  (how?)

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- 6: delete nodes in  $B$  from the graph

$b$  will be an even node, and it has unexamined neighbours.



### Algorithm 51 $\text{contract}(i, j)$

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- 5: form a circular double linked list of nodes in  $B$
- 6: delete nodes in  $B$  from the graph

Every node that was adjacent to a node  
in  $B$  is now adjacent to  $b$

### Algorithm 51 $\text{contract}(i, j)$

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- 6: delete nodes in  $B$  from the graph

Only for making a blossom expansion easier.

### Algorithm 51 $\text{contract}(i, j)$

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- 5: form a circular double linked list of nodes in  $B$
- 6: delete nodes in  $B$  from the graph

Only delete links from nodes not in  $B$  to  $B$ .  
When expanding the blossom again we can  
recreate these links in time  $\mathcal{O}(m)$ .

# Analysis

- ▶ A contraction operation can be performed in time  $\mathcal{O}(m)$ . Note, that any graph created will have at most  $m$  edges.
- ▶ The time between two contraction-operation is basically a BFS/DFS on a graph. Hence takes time  $\mathcal{O}(m)$ .
- ▶ There are at most  $n$  contractions as each contraction reduces the number of vertices.
- ▶ The expansion can trivially be done in the same time as needed for all contractions.
- ▶ An augmentation requires time  $\mathcal{O}(n)$ . There are at most  $n$  of them.
- ▶ In total the running time is at most

$$n \cdot (\mathcal{O}(mn) + \mathcal{O}(n)) = \mathcal{O}(mn^2) .$$

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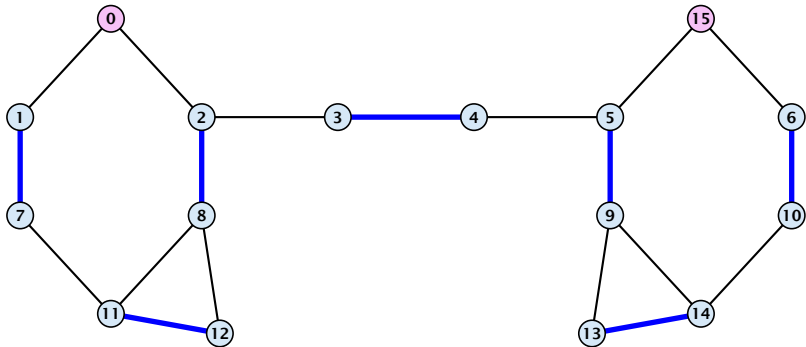


## Analysis

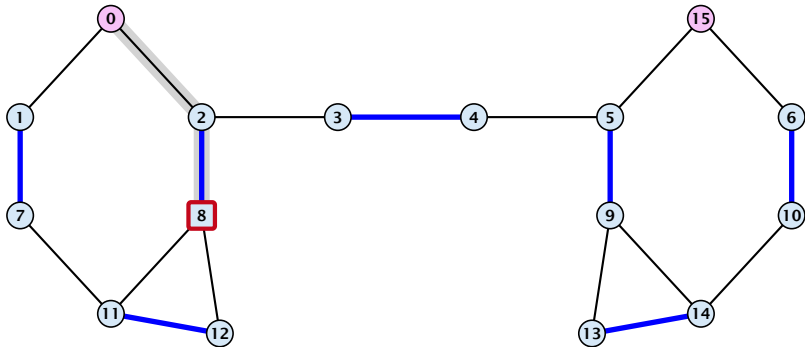
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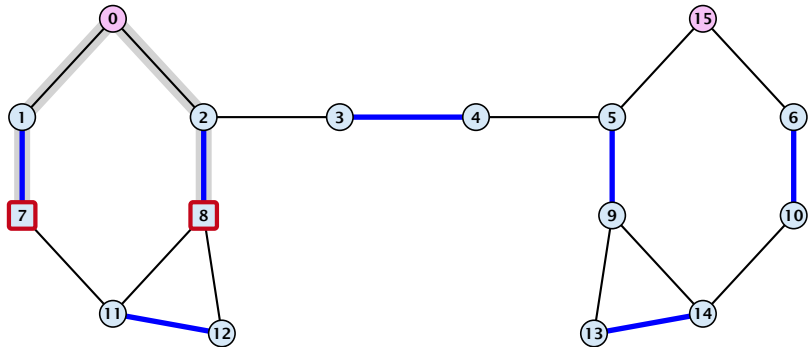
# Example: Blossom Algorithm



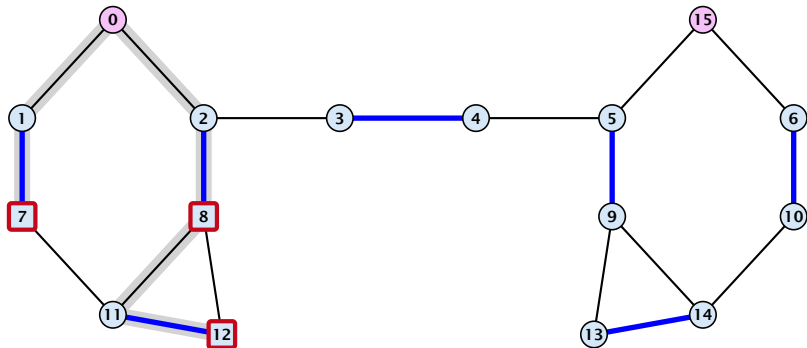
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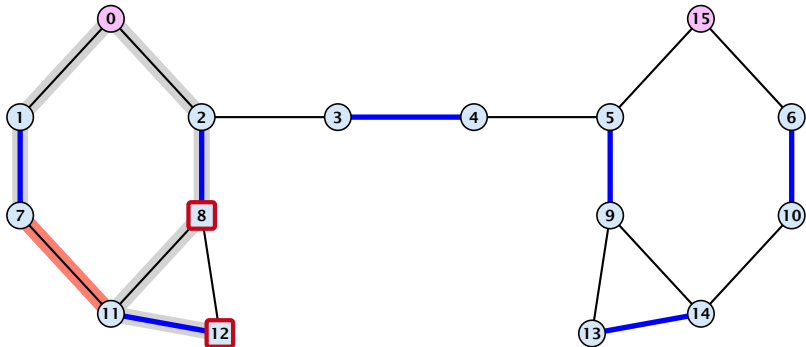
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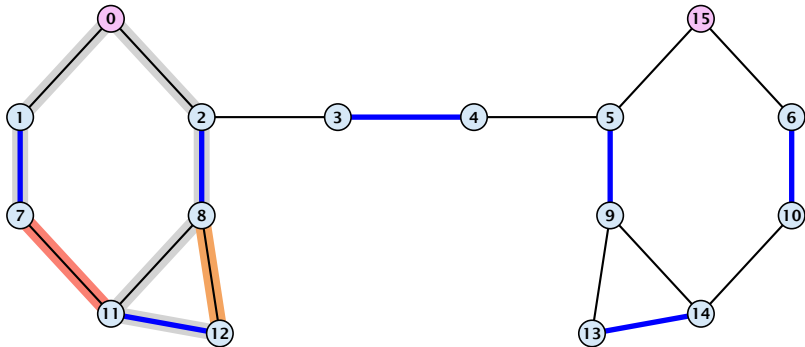
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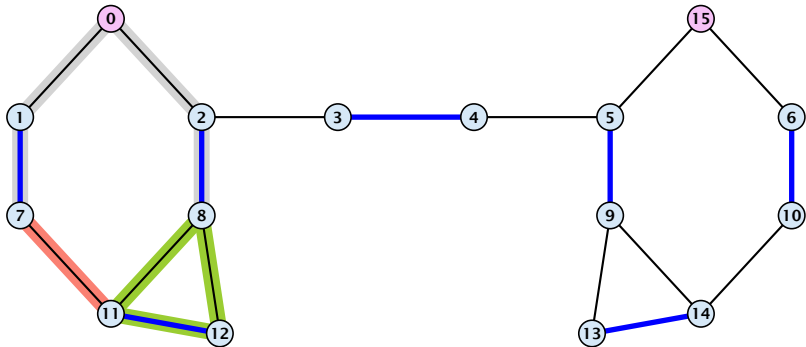
# Example: Blossom Algorithm



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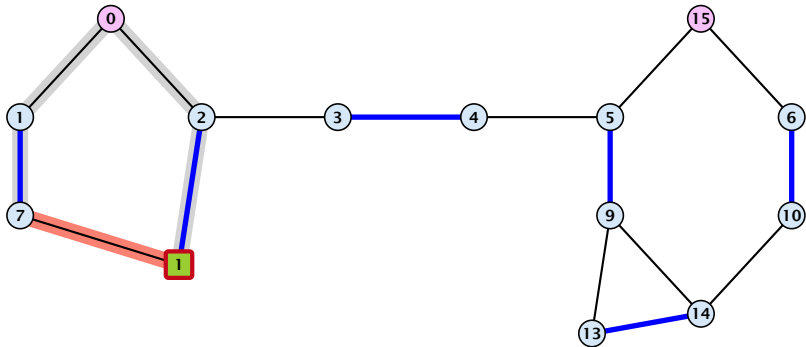


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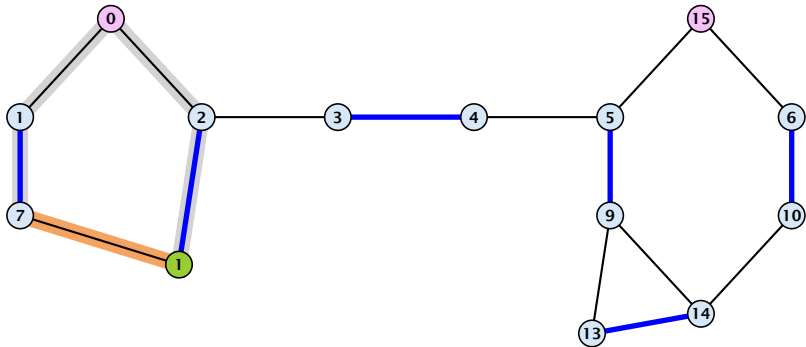




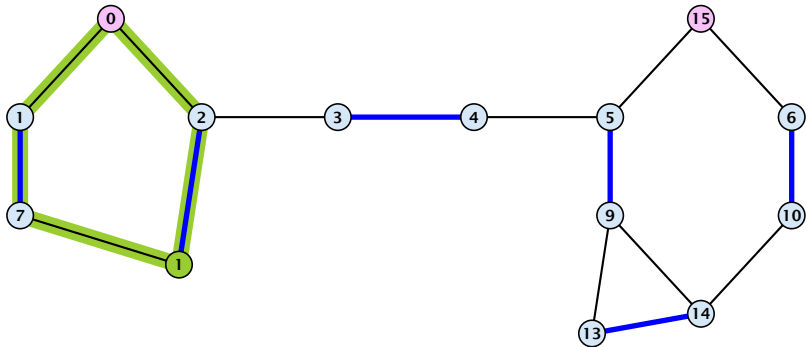
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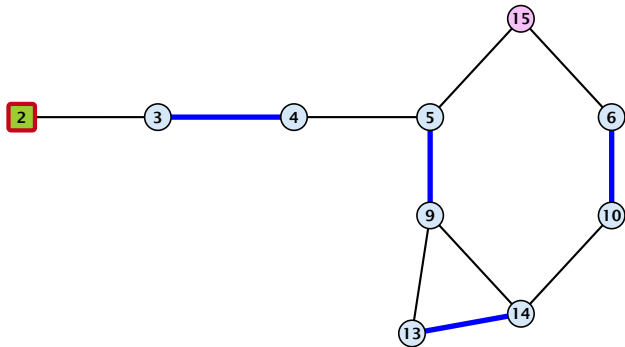
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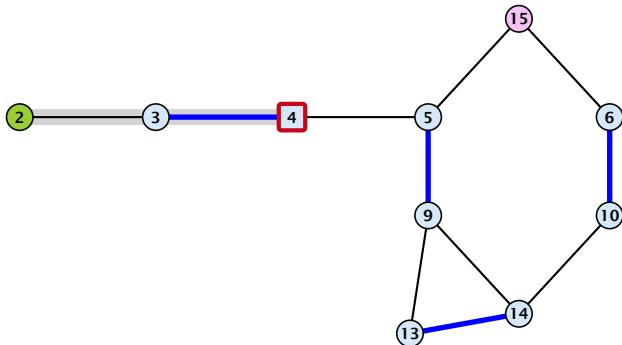
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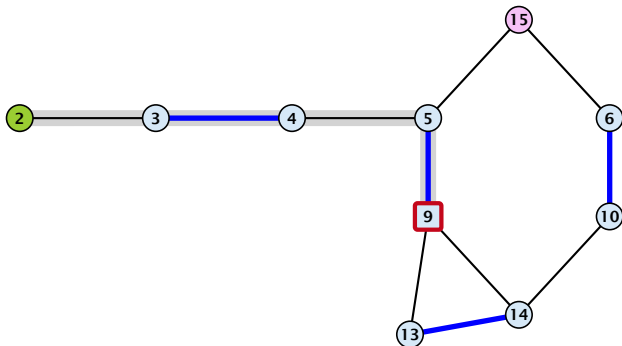
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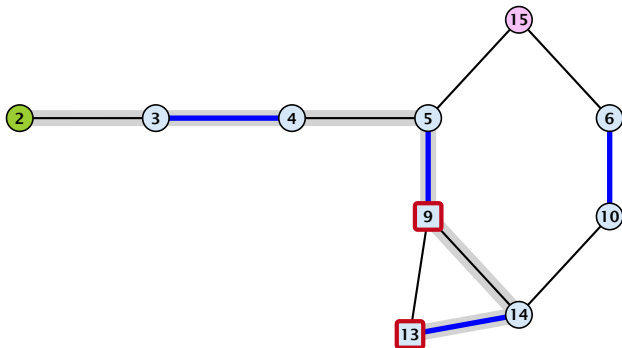
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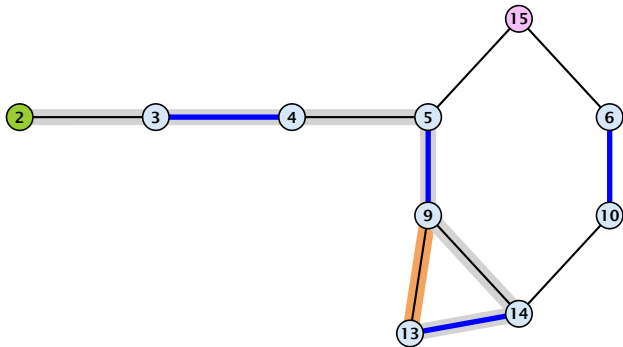
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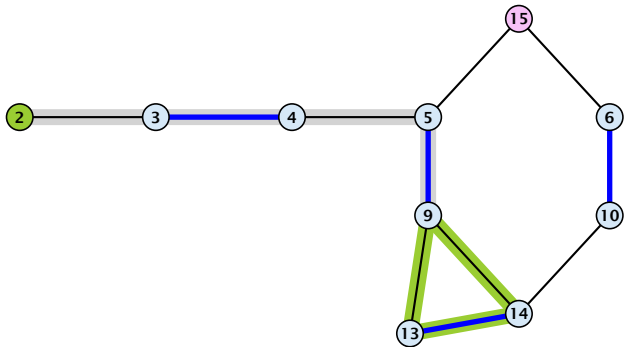


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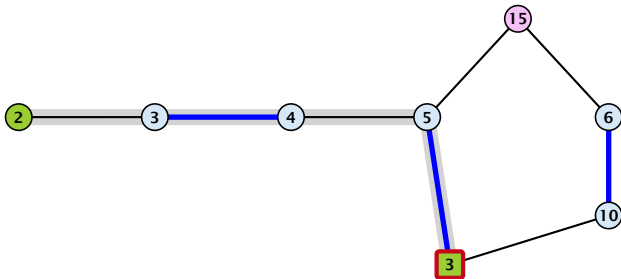




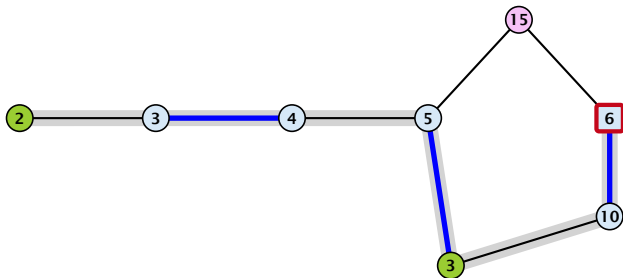
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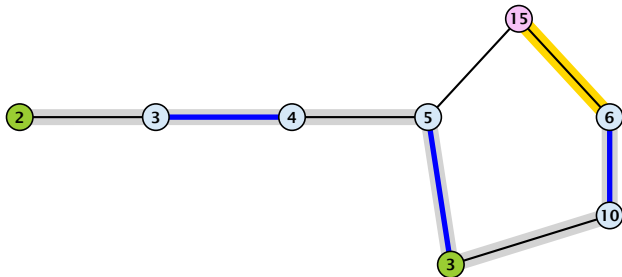
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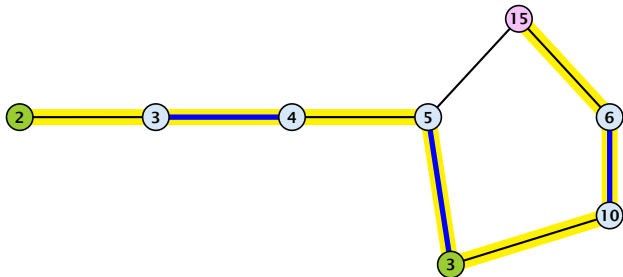
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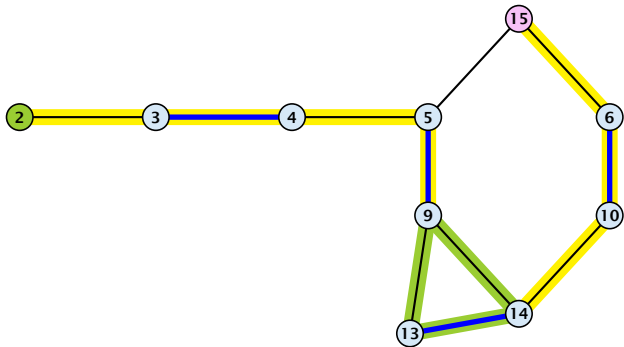
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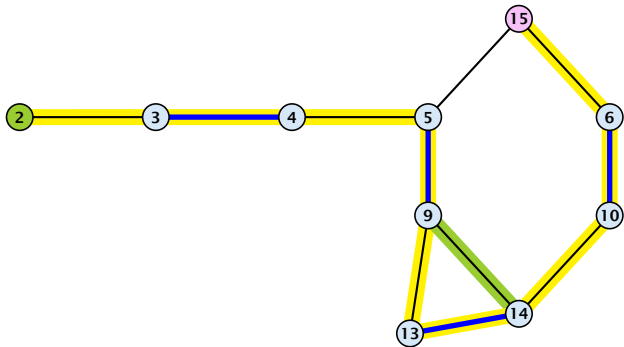
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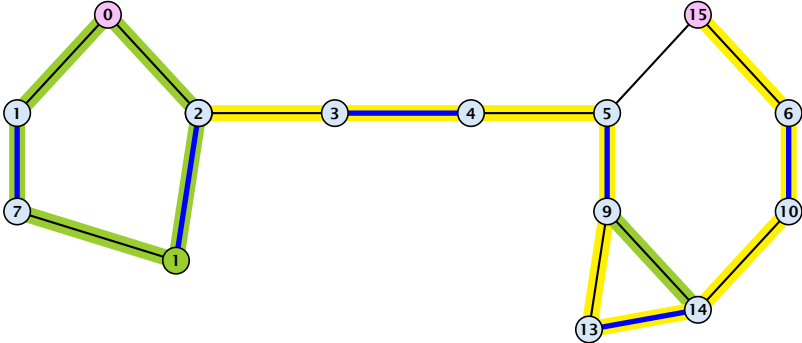
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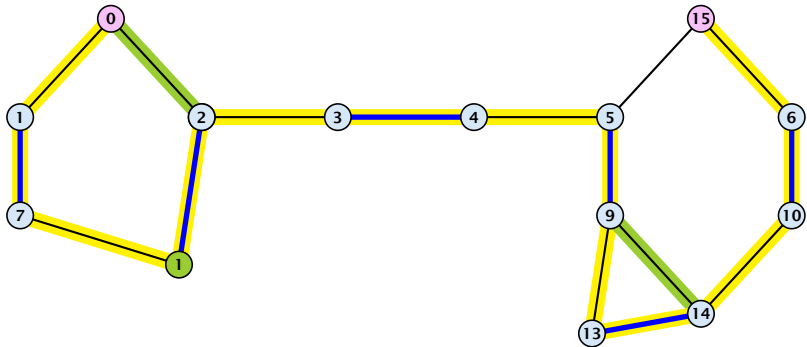


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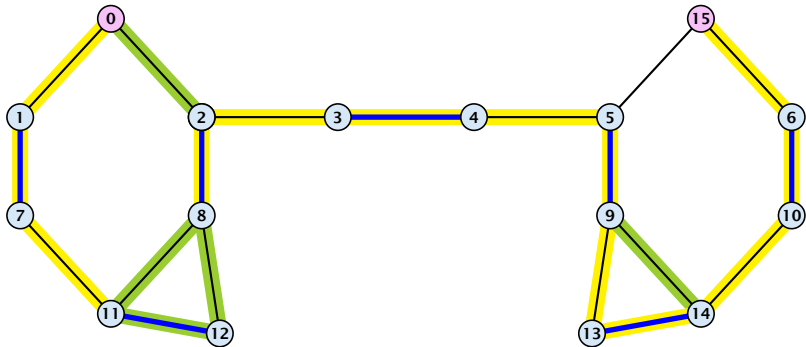




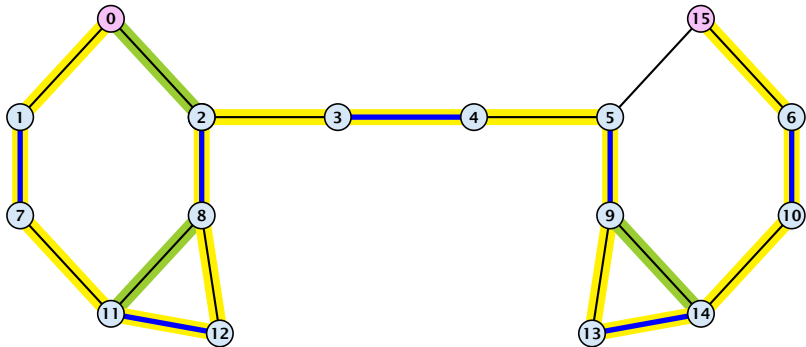
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